Categorical shadows lurking behind integral formulas for genera

(sorry for this)

Based on joint work with Mattia Cloma and Eugenio Landi
(arXiv: 1911.12035)

$\Omega^U_\bullet$ = complex cobordism ring

Ω^U_d = cobordism classes of d-dim closed (stably) complex manifolds

By definition a *genus* with values in a comm. ring R is a ring homomorphism

$\gamma : \Omega^U_\bullet \longrightarrow R$.

We'll be interested in genera

$\gamma : \Omega^U_\bullet \longrightarrow \mathbb{Q}$

This will be equivalent to ring morphisms

$$\gamma_{\mathbb{Q}} : \underset{\substack{\wr\wr \\ \mathbb{Q}[t_1, t_2, t_3 \ldots]}}{\Omega^U_\bullet \otimes \mathbb{Q}} \longrightarrow \mathbb{Q}$$

$\deg t_i = 2i$

$t_i = [\mathbb{P}^i\mathbb{C}]$

$\gamma_{\mathbb{Q}}$ is equivalent to the datum of a sequence of rational numbers, $a_i := \gamma_{\mathbb{Q}}(t_i)$

Example : The Todd genus is the genus corresponding to $a_i = 1 \quad \forall i$

If X is a compact complex manifold then

$$\underbrace{\gamma_{td}([X])}_{\text{Todd genus of } X} = \int_X td(X)$$

a certain cohomology class

Hirzebruch genus formula : for any genus γ, there exist a universal cohomology class td_γ (for any manifold X, any complex vector bundle $V \to X$ $td_\gamma(V)$ which is natural with respect to morphisms of manifolds / pullbacks of bundles :

$$td_\gamma(f^*V) = f^* td_\gamma(V) \, ; \quad td_\gamma(X) := td_\gamma(TX) \,)$$

such that

$$\gamma[X] = \int_X td_\gamma(X).$$

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i) The category of spectra as a setting for cohomology theories.

$$Top \longrightarrow Sp$$

The motto is : Spectra are to spaces as real numbers are to rational numbers.

$$(\mathrm{Top}, \times) \xrightarrow[\text{monoidal}]{(\ )_*} (\mathrm{Top}_*, \wedge) \xrightarrow[\text{monoidal}]{\Sigma^\infty} (\mathrm{Sp}, \otimes)$$

pointed topological spaces

If X is a space $\Sigma^\infty X_+$ retains the stable information on X.

It will be convenient to use the same symbol X both for X as a space and for X as a spectrum.

i) Spaces are special inside spectra

$X \longrightarrow *$
$\Delta : X \longrightarrow X \times X$

$\rightsquigarrow$

$X \longrightarrow \mathbb{S}$
$\Delta : X \to X \otimes X$

monoidal unit for Sp

(sphere spectrum $\mathbb{S}_n = S^n$)

$*$ is the terminal object in Top; it is also the monoidal unit

make X a comonoid in Top $\longrightarrow$ spaces are comonoids in spectra

ii) We consider now a monoid E in spectra

$E \otimes E \to E$

$\mathbb{S} \to E$

We will call this a ring spectrum (if it is commutative up to given coherent homotopies, we call it E_∞ - ring spectrum)

$$\underset{\underset{\text{comonoid}}{\uparrow}}{X \text{ space}}, \qquad \underset{\underset{\text{monoid}}{\uparrow}}{E \text{ ring spectrum}}$$

$[X, E]$ is a monoid

$\uparrow$ Hom set : $\pi_0 Sp(X, Y)$

$$[X, E] \times [X, E] \xrightarrow{\otimes} [X \otimes X, E \otimes E] \xrightarrow{\Delta^*, m_*} [X, E].$$

iii) Sp is an ∞- stable category:

we have homotopies between morphisms, homotopies between homotopies, and so on;

$\exists$ 0 object

Every pullback diagram is a pushout and vice-versa.

$$\begin{tikzcd} X[-1] = \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X[1] \end{tikzcd}$$

$$Sp(X, \Omega Y) = \Omega Sp(X, Y)$$

$$Y = Y[2][-2]$$

$$\Rightarrow Sp(X, Y) = Sp(X, Y[2][-2])$$

$$= Sp(X, \Omega^2 Y[2])$$

$$= \Omega^2 Sp(X, Y[2])$$

$$[X, Y] = \pi_2 Sp(X, Y[2])$$

This is an abelian group

In the particular case $Y = E, \nu$ and X is a space then we have both an abelian group structure and a multiplication on $[X, E]$. The two are compatible and $[X, E]$ is a ring.

(This is the 0-degree component of graded cohomology ring $\bigoplus_n [X, E[n]]$)

Vector bundles and their Thom spectra.

$V \to X$ a real vector bundle over a space X

$$\begin{tikzcd}
V \setminus \text{zero section} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & V \arrow[d] \\
* \arrow[r] & Th(V \to X)
\end{tikzcd}$$

It is naturally a pointed space, so its Σ^∞ is a spectrum. This is the Thom spectrum, we'll denote it by X^V.

X space $\longrightarrow$ X comonoid in Sp

$V \to X$ vector bundle $\longrightarrow$ (X, X^V)

co-monoid comodule over X

E a ring spectrum $\to$ $[X^V[n], E]$ is a module over the ring $[X, E]$, $\forall n \in \mathbb{Z}$.

All this extends from vector bundles to virtual vector bundles $V = V_1 \ominus V_2$

Not only spaces are special inside spectra but closed (compact without boundary) smooth manifolds are special within spaces seen as spectra!

Sp is monoidally closed:

$$Sp(Y \otimes X, Z) = Sp(Y, F(X, Z))$$

Def. The Alexander-Spanier dual of X is $DX := F(X, \mathbb{S})$

$$Sp(Y \otimes X, \mathbb{S}) = Sp(Y, DX)$$

$X \longmapsto DX$ is a contravariant functor

$$\mathbb{S} \overset{D}{\longmapsto} D\mathbb{S} = \mathbb{S}$$

$$X \longrightarrow \mathbb{S} \quad \rightsquigarrow \varphi_X : \mathbb{S} \longrightarrow DX$$

for any space.

Smooth manifolds are special:

$$DX = X^{-TX}.$$

We say that X is E-<u>orientable</u> if $[X^{-TX}[\dim X], E]$ is a rank 1 $[X, E]$-module. An E-orientation is a module isomorphism

$$[X^{-TX}[\dim X], E] \xleftarrow[\sigma]{\sim} [X, E].$$

Two orientations will "differ" by multiplication by an invertible element in $[X, E]$.

<u>Back to integral formulas.</u>

E-Orientations $\longrightarrow$ E-integration.

$$[X, E] \xrightarrow{\sigma} [X^{-TX}[\dim X], E]$$

$$\|$$

$$[DX[\dim X], E] \xrightarrow{\varphi^*} [\mathbb{S}[\dim X], E] = \pi_{\dim X} E.$$

Fact: an E-orientation for (stably) complex vector bundles is equivalent to a morphism of homotopy ring spectra $MU \xrightarrow{\varphi} E$. One calls φ a <u>complex orientation</u> of E.

<u>Examples</u>: i) $MU \xrightarrow{id} MU$

ii) $H\mathbb{Q}^{per}$ has a standard complex orientation $MU \xrightarrow{\varphi_{st}} H\mathbb{Q}^{per}$. The corresponding integration is the usual integration on closed complex manifolds

Let now $\varphi : MU \longrightarrow H\mathbb{Q}^{per}$ be any complex orientation. This will differ by φ_{st} by multiplication by an invertible element. Let us call this element Td_φ.

Then we have a commutative diagram, for any complex manifold X,

$$\begin{tikzcd}
{[X, MU]} \arrow[rr, "\mathrm{td}_\psi \cdot \psi_*"] \arrow[d, "\int_X^{MU}"'] & & {[X, H\mathbb{Q}^{per}]} \arrow[d, "\int_X^{H\mathbb{Q}^{per}}"] \\
{[\mathbb{S}, MU[-\dim_{\mathbb{R}} X]]} \arrow[rr, "\psi_*"] \arrow[d, phantom, "\shortparallel"] & & {[\mathbb{S}, H\mathbb{Q}[-\dim_{\mathbb{R}} X]]} \arrow[d, phantom, "\shortparallel"] \\
\Omega^U_{\dim_{\mathbb{R}} X} \arrow[rr, "\psi_*"] & & \mathbb{Q}
\end{tikzcd}$$

Taking $1 \in [X, MU]$ we get

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & \mathrm{td}_\psi \arrow[d] \\
{[X]} \arrow[r, mapsto, "\psi_*"] & \psi_*[X] = \int_X \mathrm{td}_\psi
\end{tikzcd}$$