A Skew Approach to Enrichment for Gray-Categories

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$$\begin{tikzcd} \mathrm{ITA} \arrow[r, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathrm{CA} \arrow[l, shift left=1.5] \end{tikzcd}$$

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*j.w.w. John Bourke

25 Octcober

INTRO & MOTIVATION

- Given $\mathcal{V}$ symm. monoidal cat $\Rightarrow$ $\mathcal{V}$-Cat symm. monoidal as well!

$\rightsquigarrow$ ITERATED ENRICHMENT !

- Strict higher Cats: $0\text{-}Cat := Set$
$(n+1)\text{-}Cat := (n\text{-}Cat)\text{-}Cat$

INTRO & MOTIVATION

- Given $\mathcal{V}$ symm. monoidal cat $\Rightarrow$ $\mathcal{V}$-Cat symm. monoidal as well!

$\rightsquigarrow$ ITERATED ENRICHMENT!

- Strict higher Cats: $0\text{-Cat} := \text{Set}$
  $(n+1)\text{-Cat} := (n\text{-Cat})\text{-Cat}$

- This is nice, but what about... weak higher Cats?

1^{st} PROBLEM & 1^{st} SOLUTION

- $\boxed{\text{1-d}}$ 1-Cat vs Cat : strict = weak ok
- $\boxed{\text{2-d}}$ 2-Cat vs Bicat : strict ≠ weak ... BUT any bicat ok
  is equivalent to a 2-Cat!

1st Problem & 1st Solution

- 1-d : 1-Cat vs Cat : strict $=$ weak ok
- 2-d : 2-Cat vs Bicat : strict $\neq$ weak ... BUT any bicat ok is equivalent to a 2-Cat!
- 3-d : 3-Cat vs Tricat : NOT every tricat is equivalent to a 3-Cat

1^{st} PROBLEM & 1^{st} SOLUTION

- $\boxed{1\text{-d}}$ 1-Cat vs Cat : strict = weak ok
- $\boxed{2\text{-d}}$ 2-Cat vs Bicat : strict $\neq$ weak ... BUT any bicat is equivalent to a 2-Cat ! ok
- $\boxed{3\text{-d}}$ 3-Cat vs Tricat : NOT every tricat is equivalent to a 3-Cat :(

... BUT $(2\text{-Cat}, \otimes, \mathbb{1})$-Cat are enough !

GRAY tensor product

Rmk We couldn't use the CARTESIAN product, but $\otimes$ is still monoidal !

... The Open Problem

- $\boxed{4\text{-d}}$? vs Tetracats
- AIM: We want a "nice" tensor product on Gray-Cat to go on

  enriched in $(2\text{-Cat}, \otimes, 1)$

  ↳ Is it possible?

... The Open Problem

- $\boxed{\text{4-d}}$? vs Tetracats
- AIM: We want a "nice" tensor product on Gray-Cat to go on

  enriched in $(2\text{-Cat}, \otimes, \mathbb{1})$

  ↳ Is it possible?
- PROBLEM: There is NO monoidal biclosed str. on Gray-Cat

  1. Capturing weak transformations (Crans, 1999)
  2. Interacting well with Lack's model structure on Gray-Cat (Bourke & Gurski, 2015)

LET'S LOOK AT PART 1.

- If $[\mathcal{A},\mathcal{B}]$ is an internal hom w/ 1-cells $\eta : F \Rightarrow G$ weak transf.

$\Longleftrightarrow \quad \eta : \mathcal{2} \longrightarrow [\mathcal{A},\mathcal{B}]$ Gray-funct. $\overset{\text{BICLOSED}}{\Longleftrightarrow}$ $\eta : \mathcal{A} \longrightarrow [\mathcal{2},\mathcal{B}]$

$x \longmapsto \eta_x : Fx \to Gx$

$$\begin{tikzcd} x \arrow[d, "f"'] \\ y \end{tikzcd} \longmapsto \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta_y"'] & Gy \end{tikzcd}$$

Let's look at Part 1.

- If $[A, B]$ is an internal hom w/ 1-cells $\eta : F \Rightarrow G$ weak transf.

$$\Leftrightarrow \quad \eta : \mathbf{2} \longrightarrow [A, B] \text{ Gray-funct.} \overset{\text{BICLOSED}}{\Leftrightarrow} \eta : A \longrightarrow [\mathbf{2}, B]$$

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- PROBLEM: Functoriality in A force the equality

$$A \qquad \begin{tikzcd} x \arrow[d, "f"'] \\ y \arrow[d, "g"'] \\ z \end{tikzcd} \text{ 1-cells} \qquad \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow \eta_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta_y"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\Downarrow \eta_g"] & Gy \arrow[d, "Gg"] \\ Fz \arrow[r, "\eta_z"'] & Gz \end{tikzcd} = \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "F(gf)"'] \arrow[dr, phantom, "\Downarrow \eta_{gf}"] & Gx \arrow[d, "G(gf)"] \\ Fz \arrow[r, "\eta_z"'] & Gz \end{tikzcd}$$

- This is quite unnatural

... In fact this would make weak transf.

NOT composable!

LET'S LOOK AT PART 1.

- If $[\mathcal{A},\mathcal{B}]$ is an internal hom w/ 1-cells $\eta : F \Rightarrow G$ weak transf.

$\iff$ $\eta : \mathcal{Q} \to [\mathcal{A},\mathcal{B}]$ Gray-funct. $\overset{\text{BICLOSED}}{\iff}$ $\eta : \mathcal{A} \to [\mathcal{Q},\mathcal{B}]$

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$$\begin{tikzcd} x \arrow[d, "f"'] \\ y \end{tikzcd} \longmapsto \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow \eta_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta_y"'] & Gy \end{tikzcd}$$

- PROBLEM: Functoriality in $\mathcal{A}$ force the equality

$\mathcal{A}$

$$\begin{tikzcd} x \arrow[d, "f"'] \\ y \arrow[d, "g"'] \\ z \end{tikzcd}$$

1-cells

$$\begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow \eta_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta_y"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\Downarrow \eta_g"] & Gy \arrow[d, "Gg"] \\ Fz \arrow[r, "\eta_z"'] & Gz \end{tikzcd} \overset{\eta_{g,f}}{\Rrightarrow} \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "F(gf)"'] \arrow[dr, phantom, "\Downarrow \eta_{gf}"] & Gx \arrow[d, "G(gf)"] \\ Fz \arrow[r, "\eta_z"'] & Gz \end{tikzcd}$$

We want something like an inv. 3-cell $\eta_{g,f}$

HINT TO THE SOLUTION FOR 1.

- We need a notion of weak map $\mathcal{A} \rightsquigarrow [\mathcal{I}, \mathcal{B}]$

Hint to the Solution for 1.

- We need a notion of weak map $A \rightsquigarrow [\mathcal{I}, B]$

- (Gohla, 2012) $\mathrm{Lax}(A, B)$ = Gray-Cat of pseudo maps of Gray-Cats

  $F : A \rightsquigarrow B$ consists of a 2-functor $F : A(x,y) \longrightarrow B(Fx, Fy)$

  \+ $\forall\, x \xrightarrow{f} y \xrightarrow{g} z$, an invertible 2-cell
$$\begin{tikzcd} Fx \arrow[r, bend left, "Ff"] \arrow[rr, bend right, "F(gf)"'] & Fy \arrow[r, bend left, "Fg"] \arrow[d, phantom, "\Downarrow F^2_{g,f}"] & Fz \\ & {} & \end{tikzcd}$$

  Cocycle!

Hint to the solution for 1.

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  $F : A \rightsquigarrow B$ (∈) consists of a 2-functor $F : A(x,y) \longrightarrow B(Fx, Fy)$

  $+ \quad \forall\, x \xrightarrow{f} y \xrightarrow{g} z$, an invertible 2-cell

$$\begin{tikzcd} & Fy \arrow[dr, "Fg"] \arrow[d, phantom, "\Downarrow F^2_{g,f}"] & \\ Fx \arrow[ur, "Ff"] \arrow[rr, bend right, "F(gf)"'] & {} & Fz \end{tikzcd}$$

Cocycle!

- WARNING: It's unlikely that we could find a representing tensor !

  ↳ In general cats of weak maps are poorly behaved...

If Not Monoidal ... What?

- AIM: Structure on Gray-Cat encoding a notion of weak map $A \rightsquigarrow B$ together with a tensor product $A \otimes B$ & internal hom $[A,B]$ interacting well.

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- Solution: CLOSED SKEW MONOIDAL!

$(a \otimes b) \otimes c \xrightarrow{\alpha} a \otimes (b \otimes c)$, $i \otimes a \xrightarrow{\rho} a$,

$a \xrightarrow{\lambda} a \otimes i$ NOT inv.!

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- Solution: closed skew monoidal!

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$a \xrightarrow{\lambda} a \otimes i$ NOT inv.!

- Weak maps $\dfrac{a \rightsquigarrow b}{i \otimes a \to b \in \mathcal{C}}$

(strict maps $a \to b \in \mathcal{C}$)

- $\mathcal{C}(a \otimes b, c) \cong \mathcal{C}(a, [b,c])$

- What's the trick?

$$\mathcal{C}(a,b) \longrightarrow \mathcal{C}(i, [a,b]) \text{ NOT inv.!}$$

$\updownarrow$

$i \otimes a \to b$, i.e.

weak maps $a \rightsquigarrow b$

THE MAIN RESULT

THEOREM (Bourke & L.) There are closed skew monoidal structures:

- $(\text{Gray-Cat}, \otimes_\ell, \mathbb{1})$ with internal hom $\mathrm{Lax}(A, B)$ ⇝ LAX GRAY PRODUCT
- $(\text{Gray-Cat}, \otimes_p, \mathbb{1})$ with internal hom $\mathrm{Psd}(A, B)$ ⇝ PSEUDO GRAY PRODUCT

↪ Also symmetric!

OK... SO SKEW... WHAT?

$$\begin{tikzcd}
\text{SKEW MON.} \arrow[rr, leftrightsquigarrow] \arrow[dr, leftrightsquigarrow] & & \text{SKEW CLOSED} \arrow[dl, leftrightsquigarrow] \\
& \text{SKEW MULTI} &
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OK... SO SKEW... WHAT?

$$\begin{tikzcd} \text{SKEW MON.} \arrow[rr, leftrightarrow, squiggly] \arrow[dr, leftrightarrow, squiggly] & & \text{SKEW CLOSED} \arrow[dl, leftrightarrow, squiggly] \\ & \text{SKEW MULTI} & \end{tikzcd}$$

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- Skew Mon.? Defining $\otimes$ is hard...
- Skew Closed? $[-,-]$ easier to define! ☺

  ... BUT $L : [B,C] \longrightarrow [[A,B],[A,C]]$ hard ☹

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- Skew Mon.? Defining $\otimes$ is hard...
- Skew Closed? $[-,-]$ easier to define! :)
  ... BUT $L : [B,C] \to [[A,B],[A,C]]$ hard :(
- Skew Multicategories!

PROBLEM: Do we have to define n-ary multimaps $\forall n \in \mathbb{N}$?!

see L.'s PhD Thesis (Bourke & L.) Nope! The 4-ary structure is enough!

Skew Multicategories

(Bourke & Lack, 2018)

A skew multicategory $\mathbb{C}$ consists of:

- objects $\mathbb{C}_0$
- loose n-multimaps $\mathbb{C}^l_n(a_1,\dots,a_n;b)$
- nullary maps $\mathbb{C}^l_0(-;a)$

Skew Multicategories

(Bourke & Lack, 2018)

A skew multicategory $\mathbb{C}$ consists of :

- objects $\mathbb{C}_0$
- tight/loose n-multimaps $\mathbb{C}^t_n(a_1,\dots,a_n;b) \subseteq \mathbb{C}^l_n(a_1,\dots,a_n;b)$
- $1_a \in \mathbb{C}^t_1(a;a)$ identity
- nullary maps $\mathbb{C}^l_0(-;a)$

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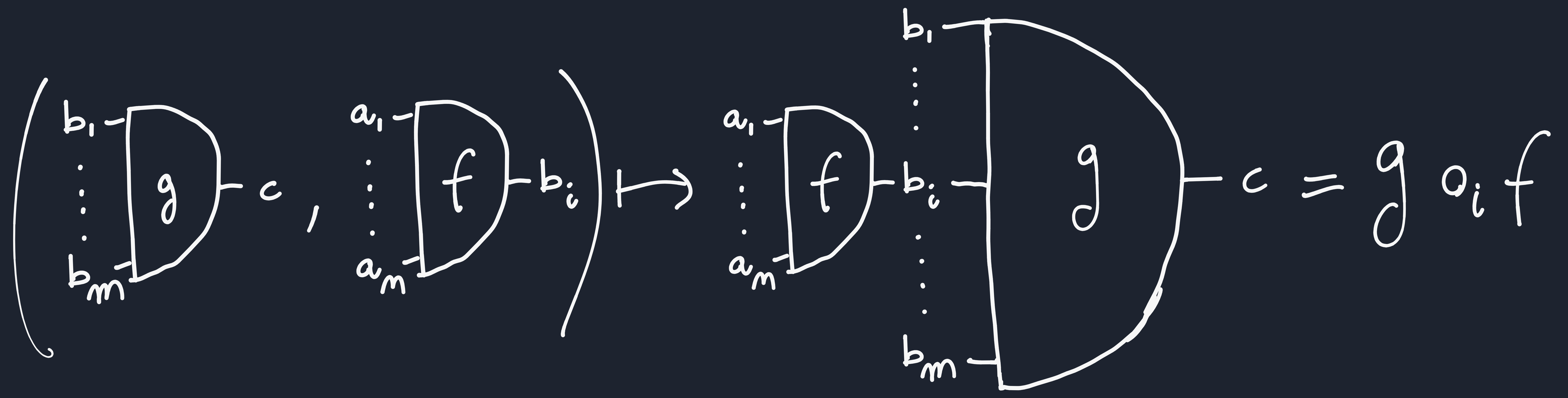
- substitutions $\circ_i : \mathbb{C}_m^{l}(\bar{b};c) \times \mathbb{C}_n^{l}(\bar{a};b_i) \longrightarrow \mathbb{C}_{n+m-1}^{l}(\bar{b}_{<i},\bar{a},\bar{b}_{>i};c)$



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s.t.

- unit and associativity laws
- $g \circ_i f$ tight whenever:
  - $i=1$, g and f tight
  - $i \neq 1$ and g tight

K-ARY SKEW MULTICATEGORIES

A K-ary skew multicategory $\mathbb{C}$ consists of :

- objects $\mathbb{C}_0$
- nullary maps $\mathbb{C}_0^{\ell}(-;a)$
- tight/loose n-multimaps $\mathbb{C}_n^{t}(a_1,..,a_n;b) \subseteq \mathbb{C}_n^{\ell}(a_1,..,a_n;b)$ $n \leq K$
- $1_a \in \mathbb{C}_1^{t}(a;a)$ identity
- substitutions $\circ_i : \mathbb{C}_m^{\ell}(\bar{b};c) \times \mathbb{C}_n^{\ell}(\bar{a};b_i) \longrightarrow \mathbb{C}_{n+m-1}^{\ell}(\bar{b}_{<i},\bar{a},\bar{b}_{>i};c)$ $n, m, n+m-1 \leq K$

s.t.

- unit and associativity laws
- $g \circ_i f$ tight whenever:
  - $i=1$, g and f tight
  - $i \neq 1$ and g tight

UNDERSTANDING TIGHT/LOOSE

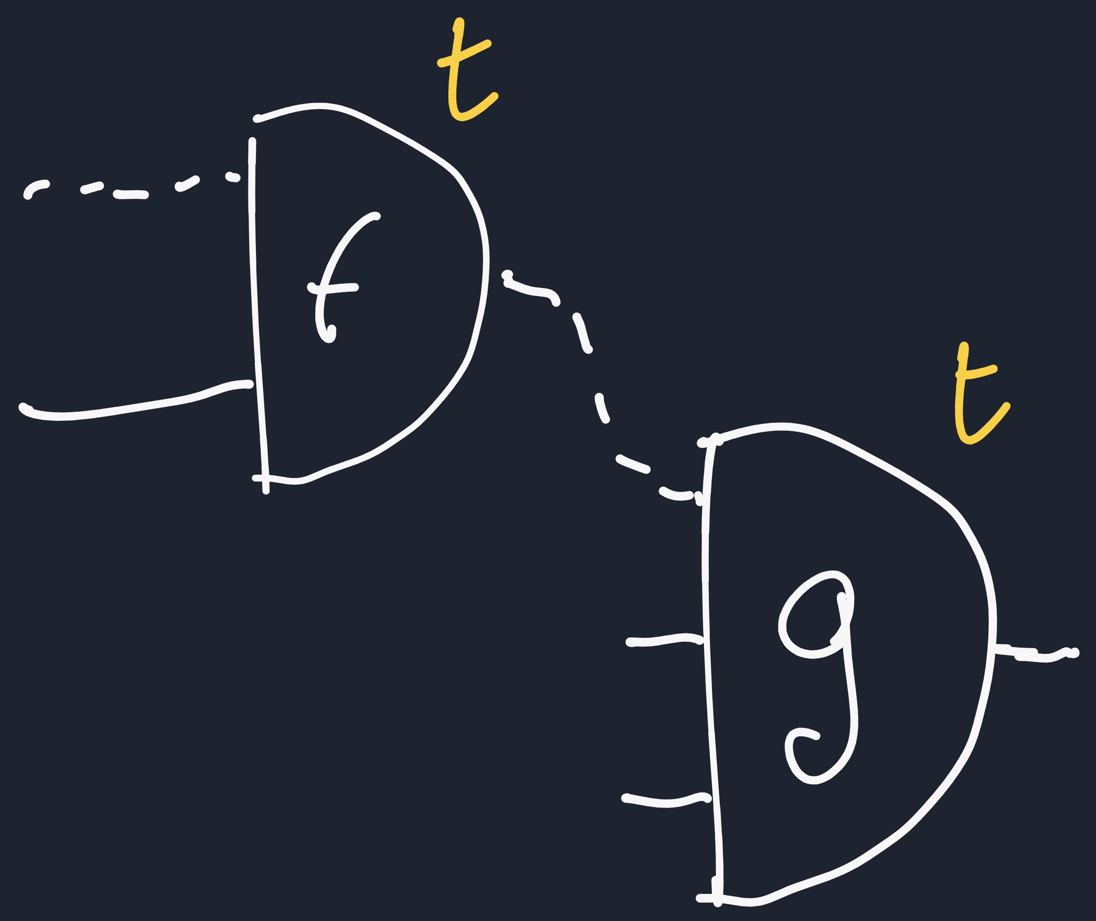
- Leading example: Cats w/ a choice of finite product

  loose n-ary $\rightsquigarrow$ funct. $A_1 \times \dots \times A_n \to B$ prod-preserving up-to-iso

  tight n-ary $\rightsquigarrow$ " $A_1 \times \dots \times A_n \to B$ " "

  & STRICTLY in A_1

- Tight ~ strict in the 1^{st} variable



= TIGHT

= LOOSE

↳ All nullary maps have to be LOOSE

THE SKEW MULTICATS $\mathbb{L}$AX/$\mathbb{P}$SD

- $\boxed{\text{0-ary}}$ $\bullet \xrightarrow{x} \mathcal{A}$, are objects $x \in \mathcal{A}$
- $\boxed{\text{1-ary}}$ loose $\mathcal{A} \rightsquigarrow \mathcal{B}$ pseudo-maps
  tight $\mathcal{A} \to \mathcal{B}$ Gray-functors
- $\boxed{\text{n-ary}}$ "Inductively"

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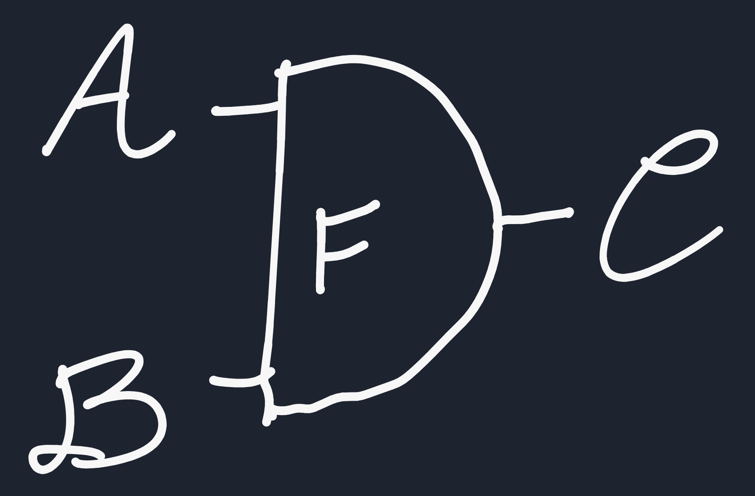

$$\leftrightsquigarrow \quad \mathcal{A} \rightsquigarrow \mathrm{Lax}(\mathcal{B}, \mathcal{C})$$

(Gray if F tight)

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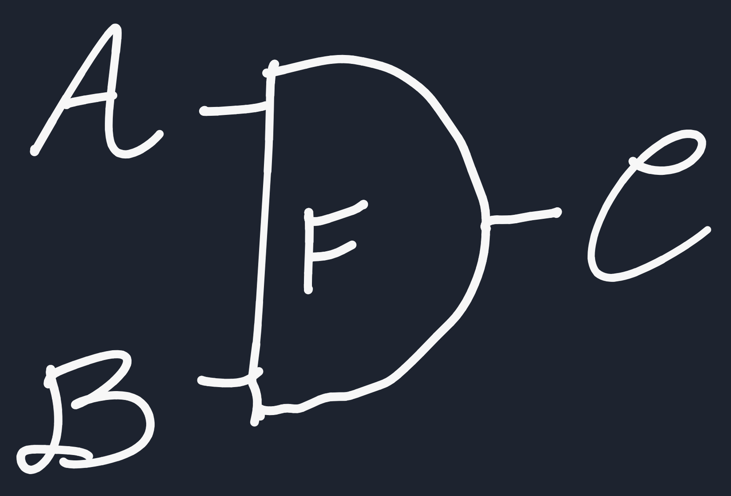

$\forall a \quad F^a : \mathcal{B} \longrightarrow \mathcal{C}$ pseudo funct

$\forall b \quad F_b : \mathcal{A} \longrightarrow \mathcal{C}$ pseudo funct

$+ \quad \forall a \xrightarrow{f} a' \quad F^f : F^a \longrightarrow F^{a'}$ lax transf

$\forall b \xrightarrow{g} b' \quad F_g : F^b \longrightarrow F^{b'}$ oplax transf

s.t. ...

If F tight
F_b Gray-functor + ...

$\rightsquigarrow \quad \mathcal{A} \rightsquigarrow \mathrm{Lax}(\mathcal{B}, \mathcal{C})$
(Gray if F tight)

THE SKEW MULTICATS Lax/Psd

- $\boxed{\text{0-ary}}$ $\bullet \xrightarrow{x} A$, are objects $x \in A$
- $\boxed{\text{1-ary}}$ loose $A \rightsquigarrow B$ pseudo-maps
  tight $A \longrightarrow B$ Gray-functors
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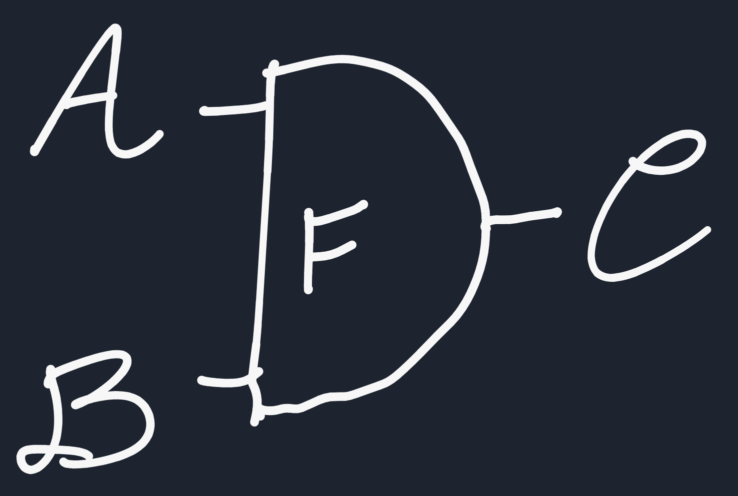

$\forall a$ $F^a : B \longrightarrow C$ pseudo funct

$\forall b$ $F_b : A \longrightarrow C$ pseudo funct

\+ $\forall a \xrightarrow{f} a'$ $F^f : F^a \longrightarrow F^{a'}$ pseudo transf

$\forall b \xrightarrow{g} b'$ $F_g : F^b \longrightarrow F^{b'}$ pseudo transf

s.t. ...

If F tight
F_b Gray-functor + ...

$\leftrightsquigarrow$ $A \rightsquigarrow Psd(B, C)$
(Gray if F tight)

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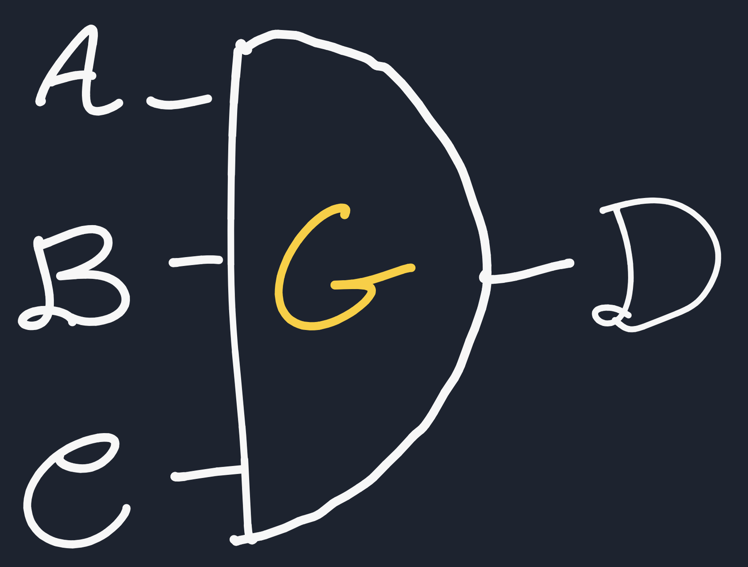

loose/tight ternary

$\forall a$ $G^a : B, C \to D$ loose binary
$\forall b$ $G_b : A, C \to D$ loose/tight binary
$\forall c$ $G_c : A, B \to D$ loose/tight binary

+ $\forall a \xrightarrow{f} a', b \xrightarrow{g} b', c \xrightarrow{h} c'$
a 3-cell $(h|g|f)$

s.t. ...

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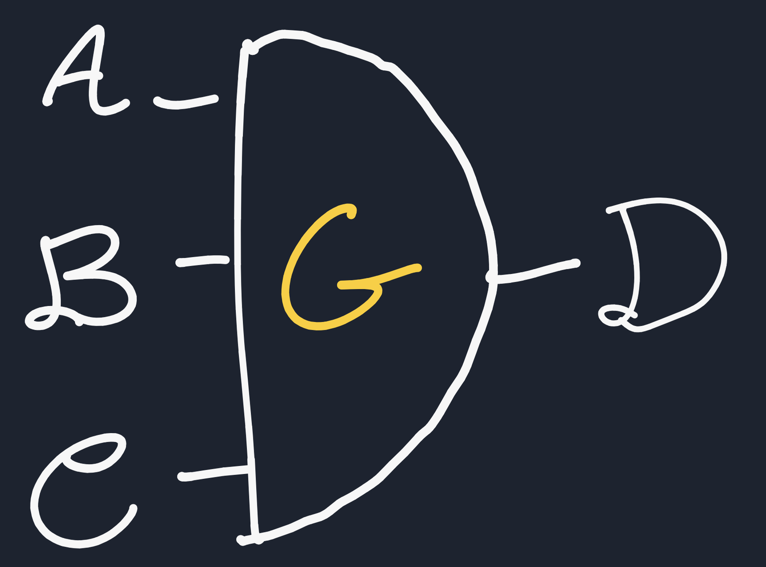

loose/tight ternary

PSEUDO

$\forall a$ $G^a : \mathcal{B}, \mathcal{C} \to \mathcal{D}$ loose^p binary

$\forall b$ $G_b : \mathcal{A}, \mathcal{C} \to \mathcal{D}$ loose/tight^p binary

$\forall c$ $G_c : \mathcal{A}, \mathcal{B} \to \mathcal{D}$ loose/tight^p binary

+ $\forall$ $a \xrightarrow{f} a'$, $b \xrightarrow{g} b'$, $c \xrightarrow{h} c'$

a inv. 3-cell $(h|g|f)$

s.t. ...

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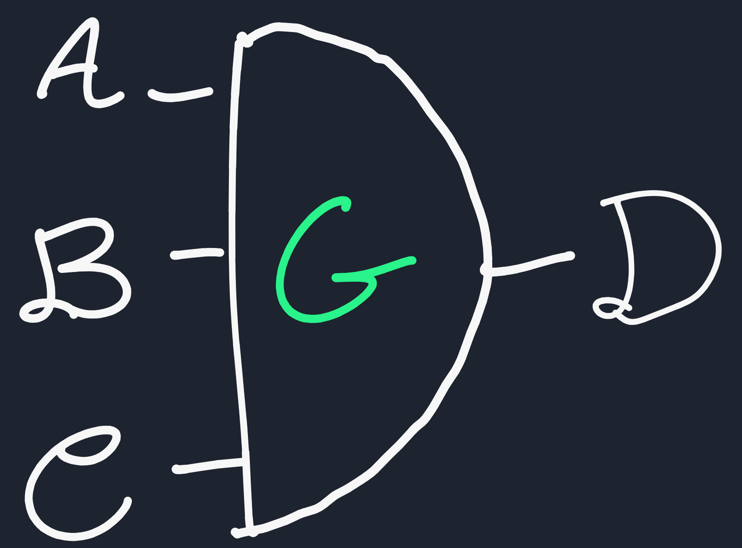
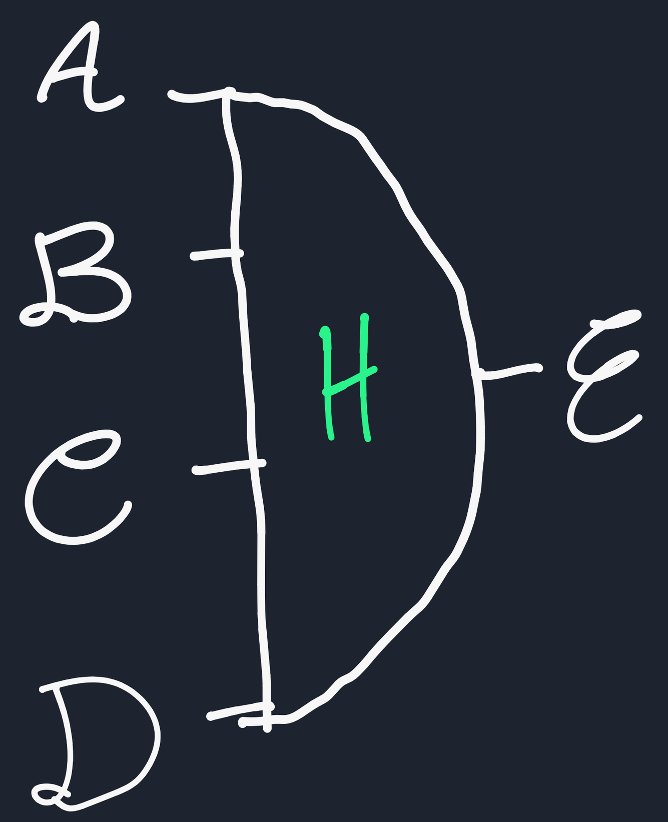
- $\boxed{n\text{-ary}}$ "Inductively"
  - F binary $\leftrightsquigarrow$ $F^a : B \to C$, $F_b : A \to C$ unary + ... s.t. ...
  - 

    $\leftrightsquigarrow$ $G^a : B, C \to D$, $G_b : A, C \to D$, $G_c : A, B \to D$ bim. + ... s.t. ...
  - 

    $\leftrightsquigarrow$ $H^a : B, C, D \to E$, $H_b : A, C, D \to E$, $H_c : A, B, D \to E$, $H_d : A, B, C \to E$ term. s.t. ...

Properties of $\otimes_p$

Follow up

- right normal, i.e. $\forall A \in \text{Gray-Cat} \quad A \otimes_p \mathbb{1} \cong A$

- weak maps $\dfrac{A \rightsquigarrow B}{\mathbb{1} \otimes_p A \to B}$ $\equiv$ Gohla's pseudo maps

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- weak maps $\dfrac{A \rightsquigarrow B}{\mathbb{1} \otimes_p A \to B} \equiv$ Gohla's pseudo maps
- symmetric (!) skew monoidal $\rightsquigarrow$ captures $\dfrac{\mathcal{X} \rightsquigarrow \mathrm{Psd}(A, B)}{A \rightsquigarrow \mathrm{Psd}(\mathcal{X}, B)}$

useful to have a "k-ary"

[continuation of L.'s PhD]

PROPERTIES OF $\otimes_p$

Follow up

- right normal, i.e. $\forall A \in \text{Gray-Cat} \quad A \otimes_p 1 \cong A$
- weak maps $\dfrac{A \rightsquigarrow B}{I \otimes_p A \to B} \equiv$ Gohla's pseudo maps
- symmetric (!) skew monoidal $\rightsquigarrow$ captures $\dfrac{\mathcal{X} \rightsquigarrow Psd(A,B)}{A \rightsquigarrow Psd(\mathcal{X},B)}$

useful to have a "k-ary" [continuation of L.'s PhD]

PROBLEM

- NOT homotopically well behaved
  - ↳ $\otimes_p$ does <u>not</u> preserve cofibrant obj.

[Lack's model str. on Gray-Cat]

THE SHARP TENSOR PRODUCT $\otimes_\#$

THEOREM (Bourke & L.) There is a closed skew monoidal structure $(\text{Gray-Cat}, \otimes_\#, \mathbb{1})$ with internal hom $\mathrm{Psd}_s(A, B)$

strict Gray-functors

PAPER

The Sharp Tensor Product $\otimes_\#$

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Paper

Properties of $\otimes_\#$

- right and left normal $A \otimes_\# \mathbb{1} \cong A \cong \mathbb{1} \otimes_\# A$
- NOT symmetric ...but...

Follow up

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FOLLOW UP !

- NOT symmetric ...but... • homotopically well behaved

  - $\otimes_\#$ preserves cofibrant obj. & weak eq. between them, $\mathbb{1}$ cofibrant
  - $- \otimes_\# A \dashv \mathrm{Psd}(A,-)$ Quillen adj.

$\leadsto$ induces a skew mon. closed str. on $\mathrm{ho}(\text{Gray-Cat})$

Semi Strict 4-Cats (not as bad as they sound)

- Memo: 2-Cat $\rightsquigarrow$ Gray-Cat

$$\begin{tikzcd} x \arrow[d, bend right, "f"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ y \arrow[d, bend right, "g"'] \arrow[d, bend left, "g'"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] \\ z \end{tikzcd}$$

$$\begin{array}{c}(\beta \circ f') \cdot (g \circ \alpha) \\ \| \\ (g' \circ \alpha) \cdot (\beta \circ f)\end{array} \quad \rightsquigarrow \quad \begin{tikzcd} gf \arrow[r, "g \circ \alpha"] \arrow[d, "\beta \circ f"'] \arrow[dr, phantom, "\Downarrow \beta : \alpha"] & gf' \arrow[d, "\beta \circ f'"] \\ g'f \arrow[r, "g' \circ \alpha"'] & g'f' \end{tikzcd}$$

Interchange

INVERTIBLE 3-CELL

satisfying axioms

(G1), (G2), (G3), (G4)

SEMI STRICT 4-CATS (NOT AS BAD AS THEY SOUND)

• MEMO: 2-Cat $\rightsquigarrow$ Gray-Cat

$$\begin{tikzcd} & x \arrow[dl, phantom] \\ \end{tikzcd}$$

$f \underset{\alpha}{\Rightarrow} f' : x \to y$, $\quad g \underset{\beta}{\Rightarrow} g' : y \to z$

$$(\beta \circ f') \cdot (g \circ \alpha) = (g' \circ \alpha) \cdot (\beta \circ f) \quad \rightsquigarrow \quad \begin{tikzcd} gf \arrow[r, "g\circ\alpha"] \arrow[d, "\beta\circ f"'] \arrow[dr, phantom, "\Downarrow \beta{:}\alpha"] & gf' \arrow[d, "\beta\circ f'"] \\ g'f \arrow[r, "g'\circ\alpha"'] & g'f' \end{tikzcd}$$

Interchange

INVERTIBLE 3-CELL satisfying axioms (G1), (G2), (G3), (G4)

• IDEA: Gray-Cat $\rightsquigarrow$ Semi-strict 4-cat $:= (\text{Gray-Cat}, \otimes_p, \mathbb{1})$-Cat

Axioms (G1) — (G4) $\rightsquigarrow$ INVERTIBLE 4-CELLS (G1) — (G4)

true ... modulo cocycles

What do we bring home

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BUT If left repr. and/or closed $\Rightarrow$ Enough n-ary up to $n \leq 4$

The End

Thanks for listening!

BONUS: COMPARISON WITH OTHER S.S. 4-CATS

- CRANS, 4-TAS: almost the same data but ...

$$\begin{tikzcd}
x \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "f''"'] \arrow[r, "f'" description] \arrow[r, phantom, shift left=3, "\Downarrow F_1"] \arrow[r, phantom, shift right=3, "\Downarrow F_2"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow G"] & z
\end{tikzcd}$$

- interchange G_F inv. 3-cell / adj. eqv.

- $$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow G_{F_1}"] & \cdot \arrow[d] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow G_{F_2}"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\quad
\begin{array}{c} \text{Equality} \\ = \\[2em] \Rrightarrow \\ \text{inv.} \\ \text{4-cell} \end{array}
\quad
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[ddr, phantom, "\Downarrow G_{(F_2 \cdot F_1)}"] & \cdot \arrow[d] \\
\cdot \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

[Bourke & L.]

NOT ENRICHED

- MIRANDA, CLOSED ENRICHED: here the hom Gray-cats must be "semi-strictly decomposable"