Topological Groupoids for Classifying Toposes

Joshua Wrigley
Università degli Studi dell'Insubria

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On topological groupoids that represent theories, arXiv:2306.16331

Main result and overview

A theory can be *represented* by topological structures, in particular certain groupoids of its models.

Can we characterise which groupoids of models contain enough information to represent a theory?

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Overview

I. Recall the definition of the topos of sheaves on a groupoid and the classifying topos of a theory.

II. Review an example representing groupoid.

III. Define elimination of parameters.

IV. Technically restate the main theorem, and sketch its providence.

V. Identify further examples of representing groupoids.

Topological groupoids

Definition

A (small) groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ consists of a diagram of sets

$$X_1 \times_{X_0} X_1 \xrightarrow{m} \underset{\circlearrowleft\, i}{X_1} \; \begin{matrix} \xrightarrow{t} \\ \xleftarrow{e} \\ \xrightarrow{s} \end{matrix} \; X_0,$$

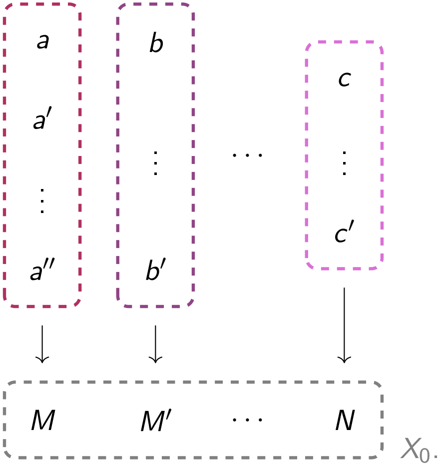
satisfying the 'obvious' equations.

A *topological groupoid* consists of topologies on X_0 and X_1 such that the above maps are continuous.

We say that $\mathbb{X}$ is *open* if s (equivalently, t) is open.

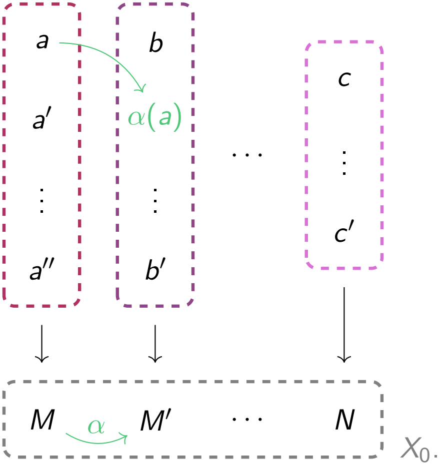
Equivariant sheaves on a groupoid

Given a groupoid $\mathbb{X}$, a discrete *bundle* on $\mathbb{X}$ consists of a map $q\colon Y \to X_0$,



Equivariant sheaves on a groupoid

Given a groupoid $\mathbb{X}$, a discrete *bundle* on $\mathbb{X}$ consists of a map $q\colon Y \to X_0$, equipped with an X_1-action $\beta\colon Y \times_{X_0} X_1 \to Y$,



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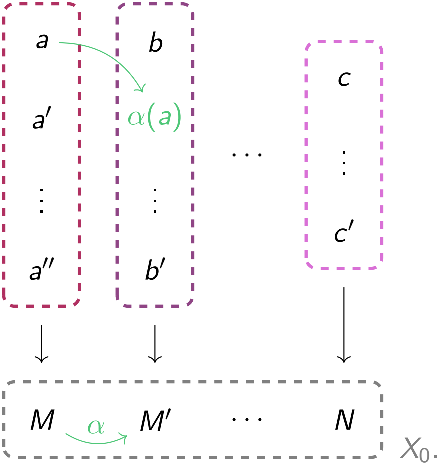

If $\mathbb{X}$ is endowed with topologies, we say that a bundle is a *sheaf* if

(i) $q\colon Y \to X_0$ is a local homeomorphism,

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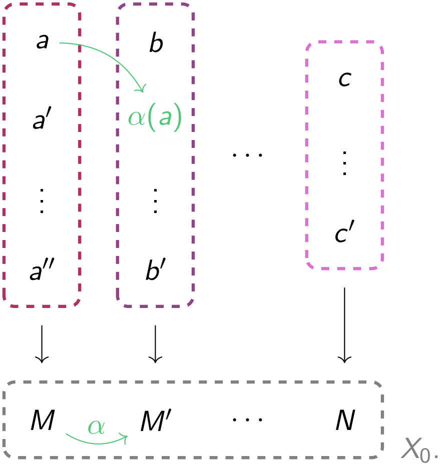

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A *morphism* of sheaves is a continuous map $f\colon Y \to Y'$ such that the following commute:

$$\begin{array}{ccc} Y \times_{X_0} X_1 & \xrightarrow{f \times_{X_0} \mathrm{id}_{X_1}} & Y' \times_{X_0} X_1 \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} \\ Y & \xrightarrow{f} & Y', \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ {\scriptstyle q}\searrow & & \swarrow{\scriptstyle q'} \\ & X_0. & \end{array}$$

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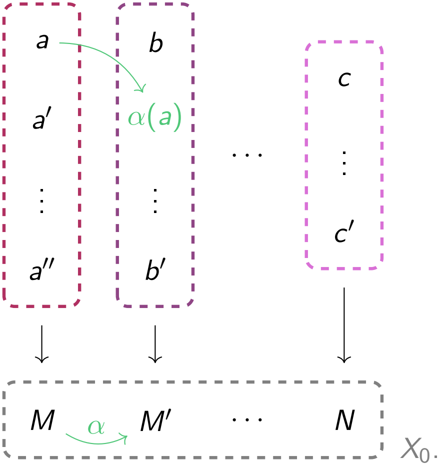

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Definition – topos of equivariant sheaves

The category of sheaves and their morphisms define a topos $\mathbf{Sh}(\mathbb{X})$.

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If a topos is ‘like’ a generalised space, then a *classifying topos* is ‘like’ a space whose points are models.

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Definition

If $\mathbb{X}$ is a (open) topological groupoid for which

$$\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}},$$

we say that $\mathbb{X}$ *represents* $\mathbb{T}$.

Indexed structures

Let $\mathbb{T}$ be a theory over a signature Σ whose set-based models are conservative.

We would expect the groupoid of *all* models to represent $\mathbb{T}$.

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Definition

Let M be a structure over a signature Σ.

Given a set $\mathfrak{K}$ of *parameters*, a $\mathfrak{K}$-*indexing* of M consists of:

(i) a subset $\mathfrak{K}' \subseteq \mathfrak{K}$,

(ii) and an *expansion* of M to the signature $\Sigma \cup \{ c_m \mid m \in \mathfrak{K}' \}$ such that M satisfies

$$\top \vdash_x \bigvee_{m \in \mathfrak{K}} x = c_m,$$

i.e. every $n \in M$ is the interpretation of some parameter $m \in \mathfrak{K}$.

Equivalently, this is a choice of partial surjection $\mathfrak{K} \rightharpoonup\!\!\!\!\rightharpoonup M$.

The groupoid of all indexed models

Let $\mathbf{Ind}(\mathfrak{K})$ denote the groupoid:

(i) whose objects are $\mathfrak{K}$-indexed models of $\mathbb{T}$,

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Theorem (Awodey–Forssell [1], [5])

Let $\mathfrak{K}$ be infinite. For suitable topologies on $\mathbf{Ind}(\mathfrak{K})$, there is an equivalence

$$\mathbf{Sh}(\mathbf{Ind}(\mathfrak{K})) \simeq \mathcal{E}_{\mathbb{T}}$$

if and only if the $\mathfrak{K}$-indexed models are *conservative* –

– if two formulae are interpreted identically, then they are $\mathbb{T}$-provably equivalent.

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This is just one example among many representing groupoids for $\mathbb{T}$.

In the following sections, we develop our characterisation.

Definable subsets of a single model

Let M be a model of $\mathbb{T}$ with an indexing $\mathfrak{K} \twoheadrightarrow M$.

(i) A *definable subset* is a subset of the form

$$[\![\vec{x} : \varphi]\!]_M = \{ \vec{n} \subseteq M \mid M \vDash \varphi(\vec{n}) \} \subseteq M^n$$

for some formula $\{ \vec{x} : \varphi \}$.

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(ii) A *definable subset with parameters* is a subset of the form

$$[\![\vec{x}, \vec{m} : \psi]\!]_M = \{ \vec{n} \subseteq M \mid M \vDash \psi(\vec{n}, \vec{m}) \} \subseteq M^n$$

for some formula $\{ \vec{x}, \vec{y} : \psi \}$ and a tuple of parameters $\vec{m} \subseteq \mathfrak{K}$.

Definables for a groupoid of models

For a groupoid $\mathbb{X}$ of $\mathbb{T}$-models, a $\mathfrak{K}$-indexing of $\mathbb{X}$ is a choice of $\mathfrak{K}$-indexing $\mathfrak{K} \twoheadrightarrow M$ for each $M \in \mathbb{X}$.

(i) A *definable* or *definable without parameters* is a subset of the form

$$[\![\vec{x} : \varphi]\!]_{\mathbb{X}} = \{ \langle \vec{n}, M \rangle \mid \vec{n} \subseteq M \in X_0,\ M \vDash \varphi(\vec{n}) \} \subseteq \coprod_{M \in X_0} M^n$$

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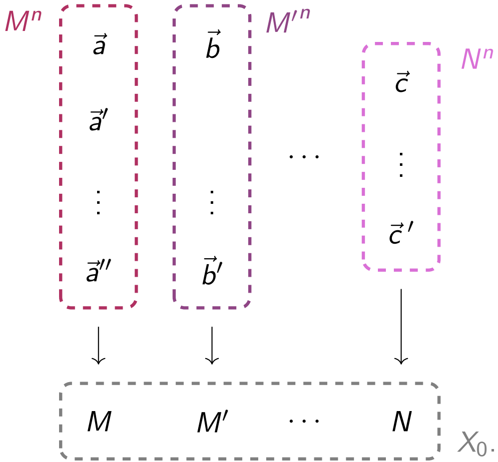
Interpreting definables and elimination of parameters

For each n, there is a bundle



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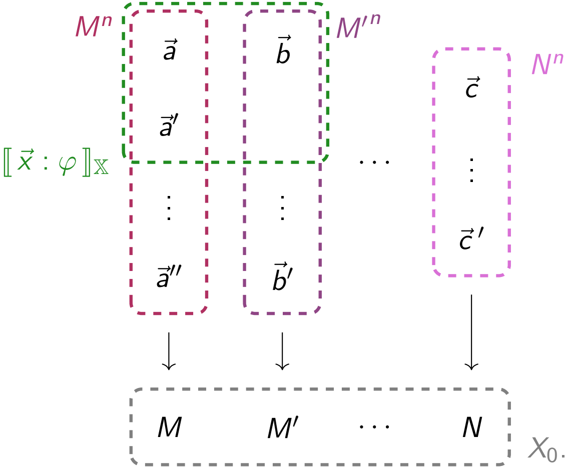

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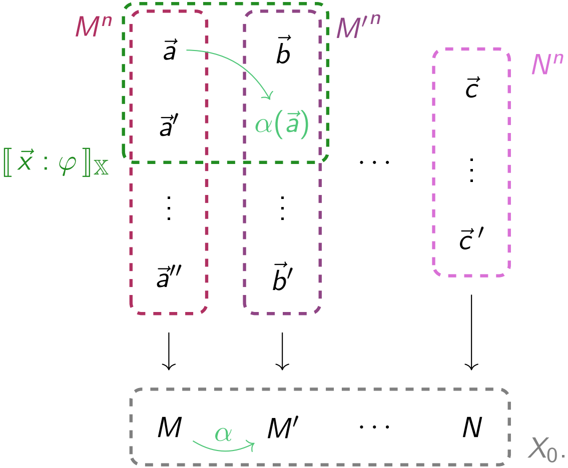

Note that $[\![\vec{x} : \varphi]\!]_{\mathbb{X}}$ is *stable* under the X_1-action.

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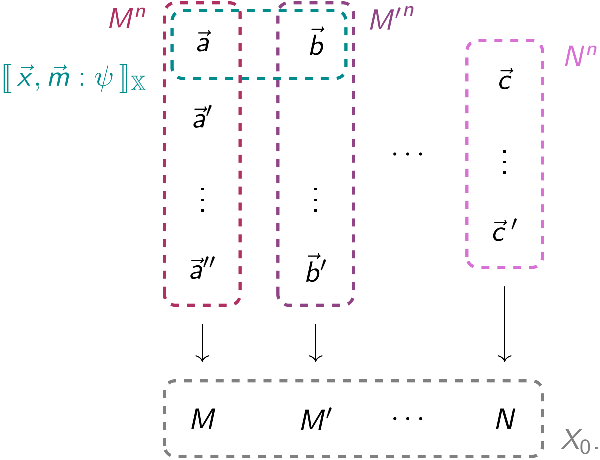
Each definable with parameters also defines a subset



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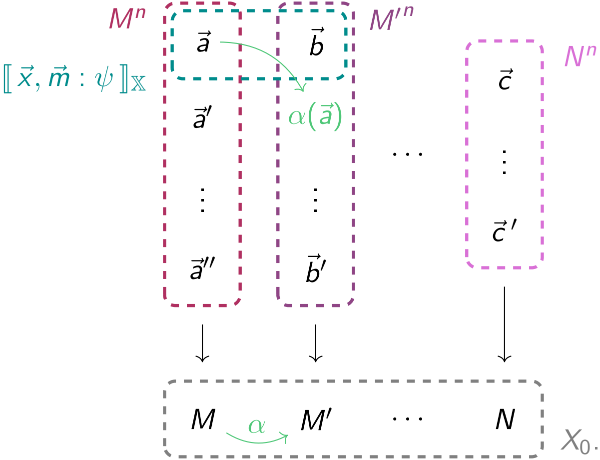
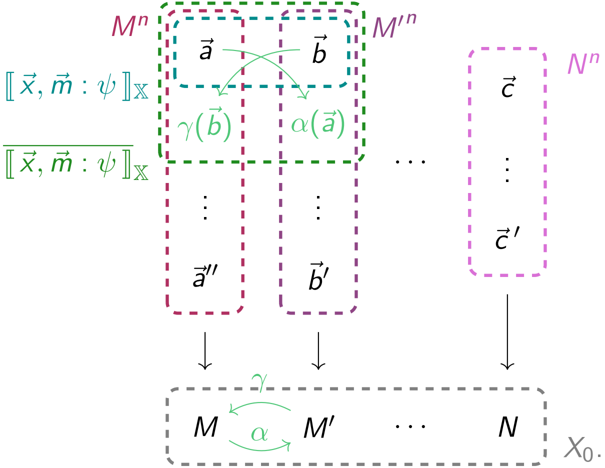


However, $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}$ is not stable under the X_1-action.

Interpreting definables and elimination of parameters

We can consider the closure of $[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}$ under the X_1-action



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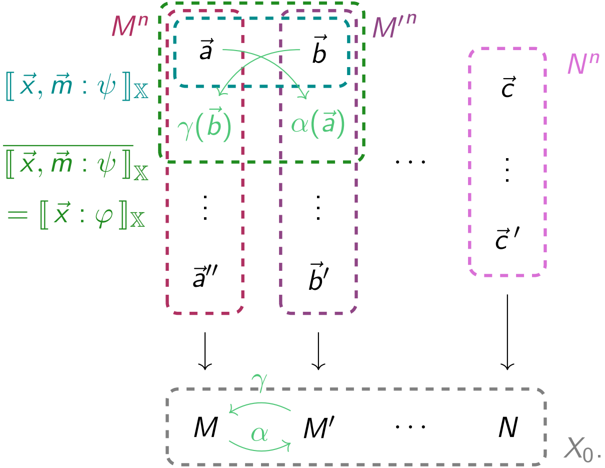

In this case, $\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}$.

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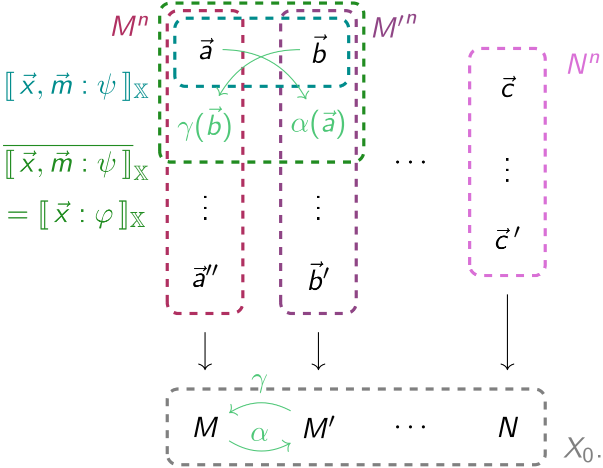

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Main Definition

Given a groupoid $\mathbb{X}$ of $\mathbb{T}$-models and an indexing $\mathfrak{K} \twoheadrightarrow \mathbb{X}$,

$\mathbb{X}$ *eliminates parameters* if, for every ψ and $\vec{m}$, there exists some *geometric* formula φ such that

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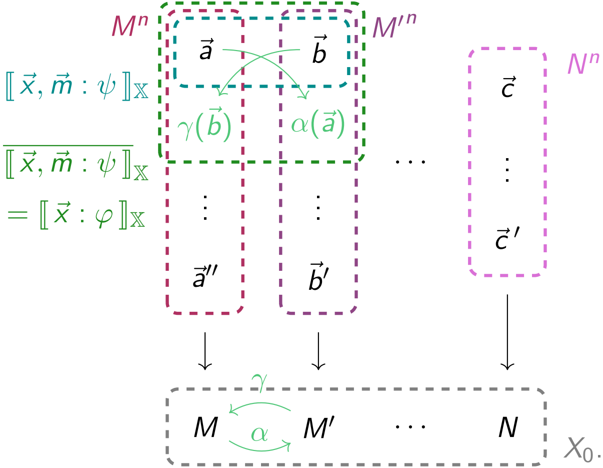

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$$\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}.$$

It suffices to check that there exists a geometric formula χ such that

$$\overline{[\![\vec{y} = \vec{m}]\!]_{\mathbb{X}}} = [\![\vec{y} : \chi]\!]_{\mathbb{X}}.$$

Classification result

Main Theorem (J.W.)

Let $\mathbb{T}$ be a geometric theory and let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a small groupoid of $\mathbb{T}$-models.

We can endow $\mathbb{X}$ with the structure of an **open** topological groupoid for which

$$\mathbf{Sh}(\mathbb{X}) \simeq \mathcal{E}_{\mathbb{T}}$$

if and only if

(i) X_0 is a conservative set –

$$[\![\vec{x} : \varphi]\!]_{\mathbb{X}} = [\![\vec{x} : \chi]\!]_{\mathbb{X}} \implies \varphi \equiv^{\mathbb{T}}_{\vec{x}} \chi,$$

(ii) there is an indexing of $\mathbb{X}$ by parameters $\mathfrak{K}$ for which $\mathbb{X}$ eliminates parameters –

$$\overline{[\![\vec{x}, \vec{m} : \psi]\!]_{\mathbb{X}}} = [\![\vec{x} : \varphi]\!]_{\mathbb{X}}.$$

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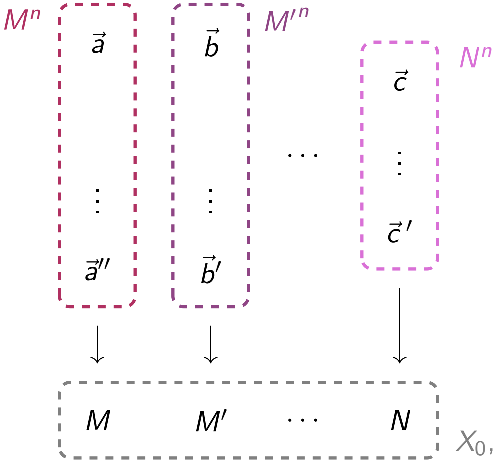

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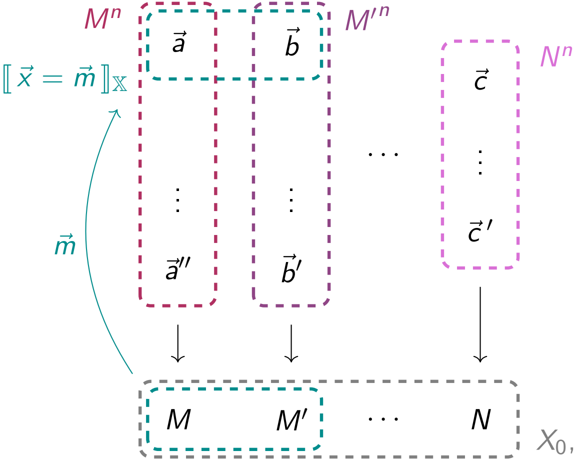

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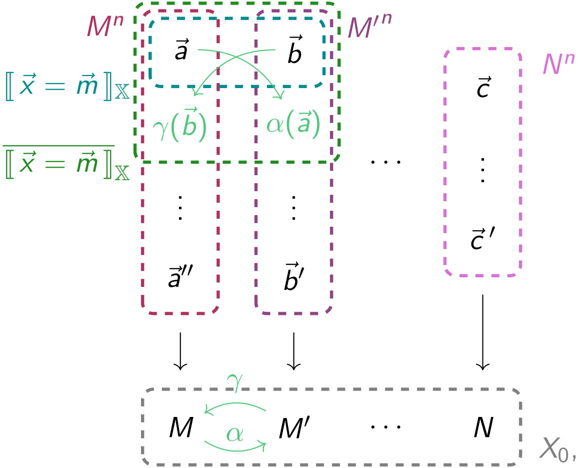

a local homeomorphism.

We index the elements of each model by the local sections with open image.

(2) A subobject $U \rightarrowtail \coprod_{M \in X_0} M^n$ in $\mathbf{Sh}(\mathbb{X})$ is a stable open subset.

In particular,

$$[\![\vec{x} : \varphi]\!]_{\mathbb{X}} \text{ and } \overline{[\![\vec{x} = \vec{m}]\!]_{\mathbb{X}}}$$

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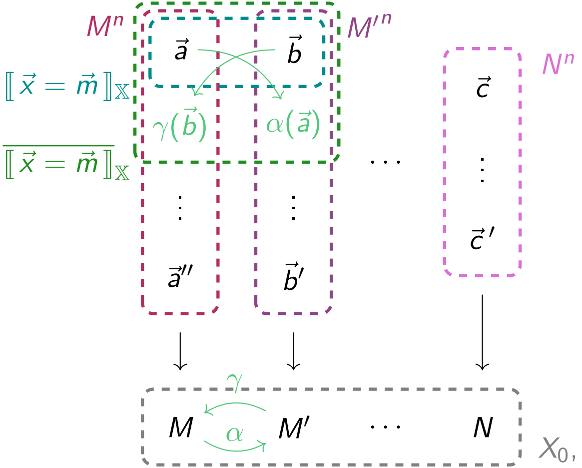

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(3) The map

$$\mathrm{Sub}_{\mathcal{E}_{\mathbb{T}}}(\{\, \vec{x} : \top \,\}) \to \mathrm{Sub}_{\mathbf{Sh}(\mathbb{X})}\left(\coprod_{M \in X_0} M^n \right)$$

$$\{\, \vec{x} : \varphi \,\} \mapsto [\![\vec{x} : \varphi]\!]_{\mathbb{X}}$$

(a) is injective if and only if $\mathbb{X}$ is conservative,

(b) and surjective if and only if $\mathbb{X}$ eliminates parameters.

Indexed and enumerated model groupoids

Proposition (cf. Awodey–Forssell [1],[5], Butz–Moerdijk [2])

(i) The groupoid of all $\mathfrak{K}$-indexed models eliminates parameters.

(ii) The groupoid of all $\mathfrak{K}$-*enumerated* –

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Indeed, for each tuple of parameters $\vec{m} \in \mathfrak{K}$,

$$\overline{[\![\, \vec{x} = \vec{m} \,]\!]_{\mathbb{X}}} = \left[\!\!\left[\vec{x} : \bigwedge_{m_i = m_j} x_i = x_j \right]\!\!\right]_{\mathbb{X}} .$$

The theory of algebraic extensions

Definition

For a fixed field K, the theory $\mathbb{T}_{(-/K)}$ *of algebraic extensions* of K is the theory

(i) with the standard axioms of a field,

(ii) constant symbols and axioms interpreting a copy of K inside any model,

(iii) and the axiom $\top \vdash \bigvee_{q \in K[x]} q(x) = 0$.

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Let $\mathrm{Sub}\left(\overline{K}\right)$ denote the groupoid of intermediate extensions

$$K \subseteq L \subseteq \overline{K}$$

and all isomorphisms between these.

We can index each $L \subseteq \overline{K}$ by the elements of $\overline{K}$.

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Proposition

The indexed groupoid $\overline{K} \twoheadrightarrow \mathrm{Sub}\left(\overline{K}\right)$ eliminates parameters and is conservative, hence

$$\mathcal{E}_{\mathbb{T}_{(-/K)}} \simeq \mathbf{Sh}\left(\mathrm{Sub}\left(\overline{K}\right)\right).$$

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(i) with the standard axioms of a field,

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(iii) and the axiom $\top \vdash \bigvee_{q \in K[x]} q(x) = 0$.

Let $\mathrm{Sub}\left(\overline{K}\right)$ denote the groupoid of intermediate extensions

$$K \subseteq L \subseteq \overline{K}$$

and all isomorphisms between these.

We can index each $L \subseteq \overline{K}$ by the elements of $\overline{K}$.

Proposition

The indexed groupoid $\overline{K} \twoheadrightarrow \mathrm{Sub}\left(\overline{K}\right)$ eliminates parameters and is conservative, hence

$$\mathcal{E}_{\mathbb{T}_{(-/K)}} \simeq \mathbf{Sh}\left(\mathrm{Sub}\left(\overline{K}\right)\right).$$

Indeed, for each tuple of parameters $\vec{a} \in \overline{K}$,

$$\overline{[\![\vec{x} = \vec{a}]\!]}_{\mathrm{Sub}(\overline{K})} = [\![\vec{x} : q_1(x_1) = 0 \wedge q_2(x_1, x_2) = 0 \wedge \cdots \wedge q_n(x_1, \dots, x_n)]\!]_{\mathrm{Sub}(\overline{K})},$$

where $q_i(x_1, \dots, x_i)$ is the minimal polynomial of a_i over $K(a_1, \dots, a_{i-1})$.

Atomic theories

A theory (with enough points) is *atomic* if and only if every (model-theoretic) type is *isolated*.

That is, for each $\vec{n} \in M$, there is a formula $\chi_{\vec{n}}$ such that $M \vDash \chi_{\vec{n}}(\vec{n})$ and, for any other tuple $\vec{n}' \in M$,

$$\vec{n}, \vec{n}' \text{ satisfy the same formulae } \iff M \vDash \chi_{\vec{n}}(\vec{n}').$$

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A model M is *ultrahomogeneous* if every partial isomorphism of finite substructures of M extends to a total automorphism

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Proposition (cf. Caramello [3])

If M is a model of an atomic theory $\mathbb{T}$, the automorphism group $\mathrm{Aut}(M)$ eliminates parameters if and only if M is ultrahomogeneous.

Indeed, if M is ultrahomogeneous, for each $\vec{m} \in M$,

$$\overline{[\![\vec{x} = \vec{m}]\!]}_{\mathbb{X}} = [\![\vec{x} : \chi_{\vec{n}}]\!]_{\mathbb{X}}.$$

Hence, $\mathcal{E}_{\mathbb{T}} \simeq \mathbf{B}\mathrm{Aut}(M)$ if and only if M is a conservative and ultrahomogeneous model.

Dense linear orders without endpoints

The theory of *dense linear orders without endpoints* $\mathbb{L}_\infty$, i.e. the theory

$$x < x \vdash_x \bot \qquad \top, \vdash_{x,y} x < y \vee x = y \vee y < x,$$

$$x < y \wedge y < z \vdash_{x,y,z} x < z, \qquad x < z \vdash_{x,z} \exists y \; x < y \wedge y < z,$$

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The rationals $(\mathbb{Q}, <)$ is an ultrahomogeneous model of $\mathbb{L}_\infty$, hence $\mathcal{E}_{\mathbb{L}_\infty} \simeq \mathbf{B}\mathrm{Aut}(\mathbb{Q})$.

The model $\mathbb{R} + \mathbb{R} \cong \{1, 2\} \times \mathbb{R}$,

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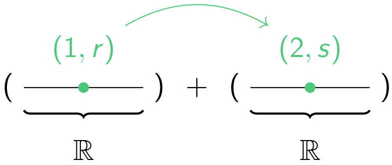

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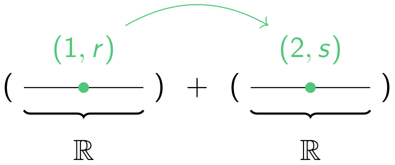

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Therefore, $\mathcal{E}_{\mathbb{L}_\infty} \not\simeq \mathbf{B}\mathrm{Aut}(\mathbb{R} + \mathbb{R})$.

In contrast, the theory of Joyal–Tierney and Dubuc [6], [4] demonstrates that

$$\mathcal{E}_{\mathbb{L}_\infty} \simeq \mathbf{B}\mathrm{Aut}(\mathbb{R} + \mathbb{R})^{\mathrm{loc}},$$

where $\mathrm{Aut}(\mathbb{R} + \mathbb{R})^{\mathrm{loc}}$ denotes the *localic* group of automorphisms of $\mathbb{R} + \mathbb{R}$.

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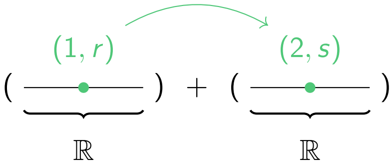

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This highlights the different flavour between the localic and topological representation of toposes.

Generic Dedekind cuts

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We note that, for all $r \in \mathbb{R}$,

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which aren't definable without parameters.

We are therefore motivated to introduce two unary predicates U_1, U_2 with interpretations $[\![\, U_1(x) \,]\!]_{\mathbb{R}+\mathbb{R}} = \{\, 1 \,\} \times \mathbb{R}$ and $[\![\, U_2(x) \,]\!]_{\mathbb{R}+\mathbb{R}} = \{\, 2 \,\} \times \mathbb{R}$.

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The group $\mathrm{Aut}(\mathbb{R}+\mathbb{R})$ eliminates parameters over this expanded signature.

Corollary

The topos $\mathbf{B}\mathrm{Aut}(\mathbb{R}+\mathbb{R})$ classifies the theory of $\mathbb{R}+\mathbb{R}$ over the expanded signature, the theory of *generic Dedekind cuts*, which is the expansion of $\mathbb{L}_\infty$ by the axioms

$$U_1(x) \wedge U_2(x) \vdash_x \bot, \qquad x < y \vdash_{x,y} U_1(x) \vee U_2(y),$$

$$\top \vdash \exists x \; U_1(x), \qquad \top \vdash \exists y \; U_2(y),$$

$$U_1(x) \wedge y < x \vdash_{x,y} U_1(y), \qquad U_2(y) \wedge y < x \vdash_{x,y} U_2(y),$$

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Thank you for listening

The preprint:

On topological groupoids that represent theories, arXiv:2306.16331

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