

Completions, Season 2, Episode 3 :

Completion under strong homotopy cokernels

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ItaCa Fest

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Starring

The category of arrows **Arr**(\mathcal{A})

objects : arrows $a: A \rightarrow A_0$ in the base category \mathcal{A}

arrows : pairs of arrows $(f, f_0): (A, a, A_0) \rightarrow (B, b, B_0)$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a \downarrow & & \downarrow b \\ A_0 & \xrightarrow{f_0} & B_0 \end{array}$$

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Previous Season

(Quite a lot of years ago) The canonical embedding

$$\mathcal{U}: \mathcal{A} \rightarrow \mathbf{Arr}(\mathcal{A}), \quad \mathcal{U}(A) = (\mathrm{id}_A: A \rightarrow A)$$

1. adds freely a factorization system (Korostenski, Tholen)
2. under suitable assumptions on \mathcal{A} , enters (in some disguised form) into several other completion processes:
 - 2.1 preregular completion
 - 2.2 reflexive coequalizer completion
 - 2.3 exact completion
 - 2.4 homological completion

(Freyd, Pitts, Carboni, Bunge, Grandis, Neeman, Rosicky, ...)

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and this is the strong homotopy cokernel of $\Gamma(a: A \rightarrow A_0)$ in $\mathbf{Arr}(\mathcal{A})$

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Nullhomotopies

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To express the notion of (strong) homotopy cokernel in a category \mathcal{B} , we need a **structure Θ of nullhomotopies** on \mathcal{B} (Grandis) :

1. for every arrow $B \xrightarrow{g} C$, a set $\Theta(g)$ of nullhomotopies on g
2. for composable arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, a map

$$f \circ g \circ h : \Theta(g) \rightarrow \Theta(f \circ g \circ h)$$

3. in such a way that, for every $\lambda \in \Theta(g)$,
 - 3.1 $(f' \circ f) \circ \lambda \circ (g \circ g') = f' \circ (f \circ \lambda \circ g) \circ g'$
 - 3.2 $\text{id}_B \circ \lambda \circ \text{id}_C = \lambda$

Notation for $\lambda \in \Theta(g)$:

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Nullhomotopies

Example : if $\mathcal{A} \subseteq \mathcal{B}$ is a subcategory, we put

$$\Theta_{\mathcal{A}}(B \xrightarrow{g} C) = \{ (g_1: B \rightarrow A, g_2: A \rightarrow C) \mid g_1 \cdot g_2 = g \text{ and } A \in \mathcal{A} \}$$

Example : in particular, if \mathcal{B} has an initial object 0 , we put

$$\Theta_0(B \xrightarrow{g} C) = \{ \lambda: B \rightarrow 0 \mid \lambda \cdot 0_C = g \}$$

Example : if \mathcal{B} is a 2-category and \mathcal{Z} an ideal of arrows in \mathcal{B} , we put

$$\Theta_{\mathcal{Z}}(B \xrightarrow{g} C) = \left\{ \text{2-cells } B \begin{array}{c} \xrightarrow{g} \\ \downarrow \lambda \\ \xrightarrow{s} \end{array} C \mid s \in \mathcal{Z} \right\}$$

Example : in particular, if \mathcal{B} has a zero object 0 and \mathcal{Z} is the ideal of zero arrows, we write Θ_0 instead of $\Theta_{\mathcal{Z}}$. This example justifies the general notation adopted for nullhomotopies

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Main example : if \mathcal{A} is additive, then $\mathbf{Arr}(\mathcal{A})$ is a 2-category ; if \mathcal{A} is any category, what remains in $\mathbf{Arr}(\mathcal{A})$ is a structure of nullhomotopies :

$$\Theta_{\Delta} \left((A, a, A_0) \xrightarrow{(f, b)} (B, b, B_0) \right) = \{ \lambda : A_0 \rightarrow B \mid a \cdot \lambda = f \text{ and } \lambda \cdot b = b_0 \}$$

A commutative diagram illustrating a nullhomotopy. It consists of two rows of objects. The top row has objects A and B , connected by a horizontal arrow f . The bottom row has objects A_0 and B_0 , connected by a horizontal arrow b_0 . A vertical arrow a points from A down to A_0 , and a vertical arrow b points from B down to B_0 . A diagonal arrow λ points from A_0 up to B . The diagram is enclosed in a light blue dashed box.

Some more examples are discussed in Episode 2. They are related to

- generalized (co)radicals and, in particular, (idempotent) (co)monads,
- prepointed categories and, in particular, multipointed categories,
- ideals of arrows seen as discrete structures of nullhomotopies.

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A commutative diagram with four nodes: A (top-left), B (top-right), A_0 (bottom-left), and B_0 (bottom-right). Arrows are: $A \xrightarrow{f} B$ (top), $A_0 \xrightarrow{f_0} B_0$ (bottom), $A \downarrow a \downarrow A_0$ (left), $B \downarrow b \downarrow B_0$ (right), and a diagonal arrow $A_0 \xrightarrow{\lambda} B$.

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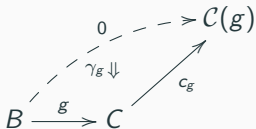
A commutative diagram illustrating a nullhomotopy. It consists of two rows of objects and three arrows. The top row has objects A and B connected by a horizontal arrow f . The bottom row has objects A_0 and B_0 connected by a horizontal arrow f_0 . A vertical arrow a points from A down to A_0 , and another vertical arrow b points from B down to B_0 . A diagonal arrow λ points from A_0 up to B . The diagram is enclosed in a large right-pointing arrow, indicating a morphism in the 2-category of arrows.

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\ominus -cokernels

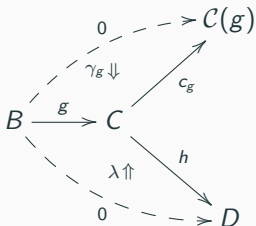
In a category with nullhomotopies (\mathcal{B}, Θ) , the Θ -**cokernel** of an arrow $g: B \rightarrow C$ is a triple $\mathcal{C}(g) \in \mathcal{B}$, $c_g: C \rightarrow \mathcal{C}(g)$, $\gamma_g \in \Theta(g \cdot c_g)$



such that, for any other triple $D \in \mathcal{B}$, $h: C \rightarrow D$, $\lambda \in \Theta(g \cdot h)$ there exists a unique arrow $h': \mathcal{C}(g) \rightarrow D$ such that

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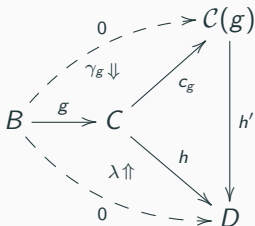
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The Θ -cokernel of $g: B \rightarrow C$ is **strong** when, in the situation

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \lambda \downarrow & & \\
 B & \xrightarrow{g} & C & \xrightarrow{c_g} & C(g) & \xrightarrow{h} & D \\
 & \searrow & \nearrow & & & & \\
 & & 0 & & & &
 \end{array}$$

if $\alpha_g \circ h = g \circ \lambda$, then there exists a unique $X \in \Theta(h)$ such that

$$c_g \circ X = \lambda$$

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if $\gamma_g \circ h = g \circ \lambda$, then there exists a unique $\lambda' \in \Theta(h)$ such that

$$c_g \circ \lambda' = \lambda$$

A couple of egocentric motivations to study Θ -(co)kernels :

1. In Episode 1, with Jacqmin, Mantovani and Metere, we used Θ -(co)kernels to give an internal unified version of the Gabriel-Ulmer exact sequence associated with a functor of pointed groupoids and of the Brown exact sequence associated with a fibration of groupoids.
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1. In Episode 1, with Jacqmin, Mantovani and Metere, we used Θ -(co)kernels to give an internal unified version of the Gabriel-Ulmer exact sequence associated with a functor of pointed groupoids and of the Brown exact sequence associated with a fibration of groupoids.
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An example from Episode 2 : given a pre-coradical

$$\begin{array}{ccc} & S & \\ \mathcal{B} & \begin{array}{c} \curvearrowright \\ \uparrow \gamma \\ \curvearrowleft \end{array} & \mathcal{B} \\ & \text{Id} & \end{array}$$

we get a nullhomotopy structure on \mathcal{B} by

$$\Theta_*(B \xrightarrow{f} C) = \{\lambda: S(B) \rightarrow C \mid \lambda_B \circ \lambda = f\}$$

For every object $B \in \mathcal{B}$, the following is the Θ -cokernel of id_B

$$\begin{array}{ccccc} & & S(B) & & \\ & \nearrow \gamma_B & & \searrow \text{id} & \\ B & \xrightarrow{\text{id}} & B & \xrightarrow{\gamma_B} & S(B) \end{array}$$

Moreover, if S is an idempotent monad and γ is its unit, then this Θ -cokernel is strong

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$$\Theta_\gamma(B \xrightarrow{g} C) = \{ \lambda : S(B) \rightarrow C \mid \gamma_B \cdot \lambda = g \}$$

For every object $B \in \mathcal{B}$, the following is the Θ_γ -cokernel of id_B :

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Main example : if \mathcal{A} has pushouts, the Θ_{Δ} -cokernel in $\mathbf{Arr}(\mathcal{A})$ of an arrow $(f, f_0): (A, a, A_0) \rightarrow (B, b, B_0)$ is a diagram of shape

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 \downarrow a & & \downarrow b & \nearrow \lambda & \downarrow c \\
 A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0
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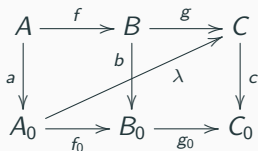
where the two following diagrams should commute

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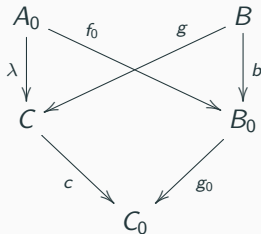
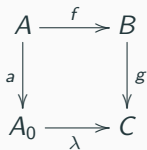
$$\begin{array}{ccccc}
 A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0 \\
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So, just replace these diagrams with the corresponding colimits and we get the strong Θ_{Δ} -cokernel.

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So, just replace the two previous diagrams with the corresponding colimits and we get the strong Θ_{Δ} -cokernel of (f, f_0) in $\mathbf{Arr}(\mathcal{A})$

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xrightarrow{a'} & A_0 +_{a,f} B \\
 \downarrow a & & \downarrow b & \nearrow f' & \downarrow [f_0, b] \\
 A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{\text{id}} & B_0
 \end{array}$$

The heart of the story

Why, if \mathcal{A} has finite colimits, the embedding

$$\Gamma: \mathcal{A} \rightarrow \mathbf{Arr}(\mathcal{A}), \quad \Gamma(A) = (\emptyset_A: \emptyset \rightarrow A)$$

is the completion of \mathcal{A} under strong Θ -cokernels ?

The heart of the story

First, the **objects** : we already observed that, for any object $a : A \rightarrow A_0$ of $\text{Arr}(\mathcal{A})$, the following is a strong Θ_Δ -cokernel

$$\begin{array}{ccccc} & & \coprod & & \\ & & \Gamma A_0 & & \\ & & \downarrow \text{id}_{\Gamma A_0} & & \\ \Gamma A & \xrightarrow{\Gamma a} & \Gamma A_0 & \xrightarrow{(\theta_a, \text{id}_{A_0})} & (A, a, A_0) \end{array}$$

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The heart of the story

Second, the **arrows** : every arrow in $\text{Arr}(\mathcal{A})$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow a & & \downarrow b \\ A_0 & \xrightarrow{f_0} & B_0 \end{array}$$

is the unique extension to the Θ_{Δ} -cokernels

$$\begin{array}{ccccc} & \Gamma A & \xrightarrow{\Gamma f} & \Gamma B & \\ & \downarrow \Gamma a & & \downarrow \Gamma b & \\ 0 & \xrightarrow{\text{id}_0} & \Gamma A_0 & \xrightarrow{\Gamma f_0} & \Gamma B_0 & \xrightarrow{\text{id}_0} 0 \\ & \downarrow (\text{id}_A, \text{id}_{A_0}) & & \downarrow (\text{id}_B, \text{id}_{B_0}) & \\ & (A, a, A_0) & \xrightarrow{(f, f_0)} & (B, b, B_0) & \end{array}$$

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$0 \xrightarrow{\text{id}_A} \Gamma A_0 \xrightarrow{\Gamma f_0} \Gamma B_0 \xleftarrow{\text{id}_B} 0$

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The heart of the story

Third, the **nullhomotopies** : every nullhomotopy in $\text{Arr}(\mathcal{A})$

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 A & \xrightarrow{f} & B \\
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 A_0 & \xrightarrow{f_0} & B_0
 \end{array}$$

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 \Gamma A & \xrightarrow{\Gamma f} & \Gamma B \\
 \downarrow \Gamma a & \nearrow \Gamma \lambda & \downarrow \Gamma b \\
 \Gamma A_0 & \xrightarrow{\Gamma f_0} & \Gamma B_0 \\
 \downarrow (0_A, \text{id}_{A_0}) & & \downarrow (0_B, \text{id}_{B_0}) \\
 (A, a, A_0) & \xrightarrow{(f, f_0)} & (B, b, B_0) \\
 \downarrow \downarrow & \nearrow \downarrow & \\
 0 & \xrightarrow{0} & 0
 \end{array}$$

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 \downarrow (0, \text{id}_{A_0}) & & \downarrow (0, \text{id}_{B_0}) & & \\
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 & & \uparrow \lambda & & \\
 & & 0 & &
 \end{array}$$

Two technical details

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In order to state correctly the main result, we need two technical details.

First : in a category with nullhomotopies (\mathcal{B}, Θ) , the reduced interchange (Grandis) is the following condition. In the situation

$$\begin{array}{ccccc} & & f & & \\ & & \curvearrowright & & \\ A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C \\ & & \curvearrowleft & & \\ & & g & & \\ & & \Theta & & \end{array}$$

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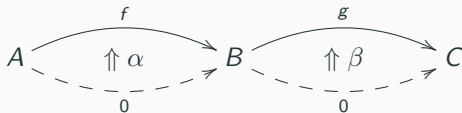
$$\begin{array}{ccccc} & & f & & \\ & & \downarrow & & \downarrow s \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ & & \downarrow g & & \downarrow t \\ & & D & & E \end{array}$$

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Main example : in $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$, the reduced interchange is true

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$$a \circ (g, g_0) = a \circ g = a \circ b \circ a = b_0 \circ d = (f, f_0) \circ d$$

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$$\alpha \circ (g, g_0) = \alpha \cdot g = \alpha \cdot b \cdot \beta = f_0 \cdot \beta = (f, f_0) \circ \beta$$

Example :

1. If \mathcal{B} is a 2-category and \mathcal{Z} is an ideal of arrows in \mathcal{B} , the reduced interchange in general is not true in $(\mathcal{B}, \Theta_{\mathcal{Z}})$. In fact, the reduced interchange would imply that for every arrow $s \in \mathcal{Z}$ there exists a unique 2-cell $s \Rightarrow s$.
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Second : Θ -strong colimits in a category with nullhomotopies (\mathcal{B}, Θ) .

1. An initial object \emptyset is Θ -strong if, for every object $B \in \mathcal{B}$, one has $\Theta(\emptyset_\circ : \emptyset \rightarrow B) = \{ \ast \}$
2. Consider the factorization through the pushout of f and g



The pushout is Θ -strong if, given $\alpha \in \Theta(x)$ and $\beta \in \Theta(y)$ such that $f \circ \alpha = g \circ \beta$, there exists a unique $[\alpha, \beta] \in \Theta([x, y])$ such that $g' \circ [\alpha, \beta] = \alpha$ and $f' \circ [\alpha, \beta] = \beta$.

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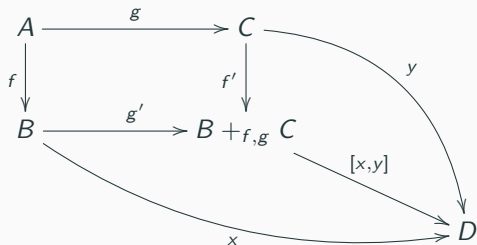
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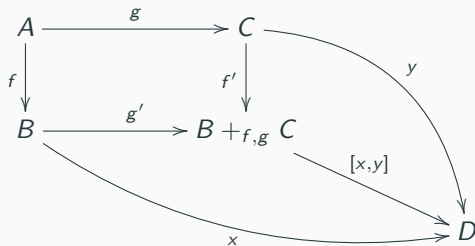
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Main example : if \mathcal{A} has finite colimits, then $\mathbf{Arr}(\mathcal{A})$ has finite colimits and they are Θ_{Δ} -strong

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From Episode 2 : why Θ -strong colimits are useful ?

1. In (\mathcal{B}, Θ) , if \mathcal{B} has strong Θ -cokernels of identity arrows and Θ -strong pushouts, then it has all Θ -cokernels and they are strong.
2. In particular, given a prepointed category

$$A \begin{array}{c} \xrightarrow{c} \\ \xrightarrow{u} \\ \xrightarrow{d} \end{array} B \quad C \rightarrow U \rightarrow D \quad U \text{ full and faithful}$$

and the structure Θ on B induced by the unit of $C \rightarrow U$ or by the counit of $U \rightarrow D$, if \mathcal{B} has pullbacks and pushouts, then they are Θ -strong and \mathcal{B} automatically has strong Θ -kernels and strong Θ -cokernels.

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Main result : If \mathcal{A} is a category with finite colimits, then the embedding

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is universal among the functors $\mathcal{F}: \mathcal{A} \rightarrow (\mathcal{B}, \Theta)$ such that

- Θ satisfies the reduced interchange,

- \mathcal{B} has strong Θ -cokernels and Θ -strong finite colimits,
- \mathcal{F} preserves finite colimits

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma} & \mathbf{Arr}(\mathcal{A}) \\ & \searrow \mathcal{F} & \downarrow \mathcal{F} \\ & & \mathcal{B} \end{array}$$

This means that, for any \mathcal{F} as above, there exists an essentially unique morphism of categories with nullhomotopies $\mathcal{F}: (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta}) \rightarrow (\mathcal{B}, \Theta)$ which preserves Θ -cokernels and finite colimits and such that $\Gamma \circ \mathcal{F} \cong \mathcal{F}$.

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is universal among the functors $\mathcal{F}: \mathcal{A} \rightarrow (\mathcal{B}, \Theta)$ such that

- Θ satisfies the reduced interchange,
- \mathcal{B} has strong Θ -cokernels and Θ -strong finite colimits,
- \mathcal{F} preserves finite colimits

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma} & \mathbf{Arr}(\mathcal{A}) \\ & \searrow \mathcal{F} & \downarrow \mathcal{F} \\ & & \mathcal{B} \end{array}$$

This means that, for any \mathcal{F} as above, there exists an essentially unique morphism of categories with nullhomotopies $\mathcal{F}: (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta}) \rightarrow (\mathcal{B}, \Theta)$ which preserves Θ -cokernels and finite colimits and such that $\Gamma \circ \mathcal{F} \cong \mathcal{F}$.

The main result

Main result : If \mathcal{A} is a category with finite colimits, then the embedding

$$\Gamma: \mathcal{A} \longrightarrow (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta})$$

is universal among the functors $\mathcal{F}: \mathcal{A} \rightarrow (\mathcal{B}, \Theta)$ such that

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This means that, for any \mathcal{F} as above, there exists an essentially unique morphism of categories with nullhomotopies $\tilde{\mathcal{F}}: (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta}) \rightarrow (\mathcal{B}, \Theta)$ which preserves Θ -cokernels and finite colimits and such that $\Gamma \circ \tilde{\mathcal{F}} \cong \mathcal{F}$.

The main result

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$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma} & \mathbf{Arr}(\mathcal{A}) \\ & \searrow \mathcal{F} & \downarrow \hat{\mathcal{F}} \\ & & \mathcal{B} \end{array}$$

This means that, for any \mathcal{F} as above, there exists an essentially unique morphism of categories with nullhomotopies $\hat{\mathcal{F}}: (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta}) \rightarrow (\mathcal{B}, \Theta)$ which preserves Θ -cokernels and finite colimits and such that $\Gamma \circ \hat{\mathcal{F}} \simeq \mathcal{F}$.

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Main result : If \mathcal{A} is a category with finite colimits, then the embedding

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- Θ satisfies the reduced interchange,
- \mathcal{B} has strong Θ -cokernels and Θ -strong finite colimits,
- \mathcal{F} preserves finite colimits

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma} & \mathbf{Arr}(\mathcal{A}) \\ & \searrow \mathcal{F} & \downarrow \widehat{\mathcal{F}} \\ & & \mathcal{B} \end{array}$$

This means that, for any \mathcal{F} as above, there exists an essentially unique morphism of categories with nullhomotopies $\widehat{\mathcal{F}}: (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta}) \rightarrow (\mathcal{B}, \Theta)$ which preserves Θ -cokernels and finite colimits and such that $\Gamma \cdot \widehat{\mathcal{F}} \simeq \mathcal{F}$

Forthcoming Episode

In the forthcoming Episode 4, with Dupont we will show that, if \mathcal{A} is an abelian category, then the bicategory of fractions

$$\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$$

is a 2-abelian bicategory.

Here Σ is a class of arrows obtained by intersection from two factorization systems constructed in $\mathbf{Arr}(\mathcal{A})$ using Θ_{Δ} -kernels and Θ_{Δ} -cokernels

See you in Palermo for PSSL 108 in September !

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