Completions, Season 2, Episode 3 : Completion under strong homotopy cokernels

Enrico M. Vitale, UCLouvain

June 16, 2023

ItaCa Fest

Table of contents

Starring

Starring

The category of arrows $\mathbf{Arr}(\mathcal{A})$

objects : arrows $a \colon A \to A_0$ in the base category $\mathcal{A}$

arrows : pairs of arrows $(f, f_0) \colon (A, a, A_0) \to (B, b, B_0)$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

commutes

Starring

The category of arrows $\mathbf{Arr}(\mathcal{A})$

objects : arrows $a\colon A \to A_0$ in the base category $\mathcal{A}$

arrows : pairs of arrows $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

commutes

Starring

The category of arrows $\mathbf{Arr}(\mathcal{A})$

objects : arrows $a\colon A \to A_0$ in the base category $\mathcal{A}$
arrows : pairs of arrows $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A_0 & \xrightarrow[f_0]{} & B_0 \end{array}$$

commutes

Previous Season

Previous Season

(Quite a lot of years ago) The canonical embedding

$$\mathcal{U} \colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,,\ \ \mathcal{U}(A) = (\mathrm{id}_A \colon A \to A)$$

1. adds freely a factorization system (Korostenski, Tholen)
2. under suitable assumptions on $\mathcal{A}$, enters (in some disguised form) into several other completion processes :
   - 2.1 preregular completion
   - 2.2 reflexive coequalizer completion
   - 2.3 exact completion
   - 2.4 homological completion

   (Freyd, Pitts, Carboni, Bunge, Grandis, Neeman, Rosicky, . . .)

Previous Season

(Quite a lot of years ago) The canonical embedding

$$\mathcal{U}\colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,,\ \ \mathcal{U}(A) = (\mathrm{id}_A\colon A \to A)$$

1. adds freely a factorization system (Korostenski, Tholen)
2. under suitable assumptions on $\mathcal{A}$, enters (in some disguised form) into several other completion processes :
   2.1 preregular completion
   2.2 reflexive coequalizer completion
   2.3 exact completion
   2.4 homological completion

   (Freyd, Pitts, Carboni, Bunge, Grandis, Neeman, Rosicky, . . .)

Previous Season

(Quite a lot of years ago) The canonical embedding

$$\mathcal{U}\colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,,\ \ \mathcal{U}(A) = (\mathrm{id}_A\colon A \to A)$$

1. adds freely a factorization system (Korostenski, Tholen)
2. under suitable assumptions on $\mathcal{A}$, enters (in some disguised form) into several other completion processes :
   2.1 preregular completion
   2.2 reflexive coequalizer completion
   2.3 exact completion
   2.4 homological completion

   (Freyd, Pitts, Carboni, Bunge, Grandis, Neeman, Rosicky, . . .)

Previous Season

(Quite a lot of years ago) The canonical embedding

$$\mathcal{U}\colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,,\ \ \mathcal{U}(A) = (\mathrm{id}_A\colon A \to A)$$

1. adds freely a factorization system (Korostenski, Tholen)
2. under suitable assumptions on $\mathcal{A}$, enters (in some disguised form) into several other completion processes :
   2.1 preregular completion
   2.2 reflexive coequalizer completion
   2.3 exact completion
   2.4 homological completion

   (Freyd, Pitts, Carboni, Bunge, Grandis, Neeman, Rosicky, . . .)

In this Episode

In this Episode

(Danger : spoil !) If the category $\mathcal{A}$ has an initial object $\emptyset$, there is another embedding

$$\Gamma\colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,,\;\; \Gamma(A) = (\emptyset_A\colon \emptyset \to A)$$

In this Episode

(Danger : spoil !) If the category $\mathcal{A}$ has an initial object $\emptyset$, there is another embedding

$$\Gamma\colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,,\ \ \Gamma(A) = (\emptyset_A\colon \emptyset \to A)$$

For a given arrow $a\colon A \to A_0$ in $\mathcal{A}$, we get the diagram

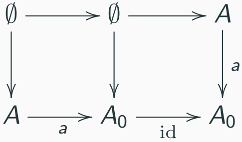
What is this ? Nothing ! But we get also the diagram

and this is the **strong homotopy cokernel** of $\Gamma(a\colon A \to A_0)$ in $\mathbf{Arr}(\mathcal{A})$

(Danger : spoil !) If the category $\mathcal{A}$ has an initial object $\emptyset$, there is another embedding

$$\Gamma \colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,,\ \ \Gamma(A) = (\emptyset_A \colon \emptyset \to A)$$

For a given arrow $a \colon A \to A_0$ in $\mathcal{A}$, we get the diagram



What is this ? Nothing ! But we get also the diagram

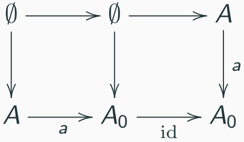
and this is the **strong homotopy cokernel** of $\Gamma(a \colon A \to A_0)$ in $\mathbf{Arr}(\mathcal{A})$

In this Episode

(Danger : spoil !) If the category $\mathcal{A}$ has an initial object $\emptyset$, there is another embedding

$$\Gamma\colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,,\ \ \Gamma(A) = (\emptyset_A\colon \emptyset \to A)$$

For a given arrow $a\colon A \to A_0$ in $\mathcal{A}$, we get the diagram



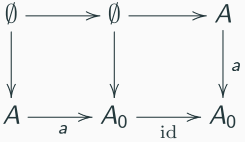
What is this ? Nothing ! But we get also the diagram

and this is the **strong homotopy cokernel** of $\Gamma(a\colon A \to A_0)$ in $\mathbf{Arr}(\mathcal{A})$

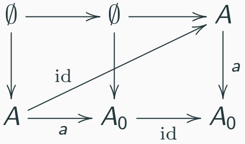
(Danger : spoil !) If the category $\mathcal{A}$ has an initial object $\emptyset$, there is another embedding

$$\Gamma\colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,,\ \ \Gamma(A) = (\emptyset_A\colon \emptyset \to A)$$

For a given arrow $a\colon A \to A_0$ in $\mathcal{A}$, we get the diagram



What is this ? Nothing ! But we get also the diagram



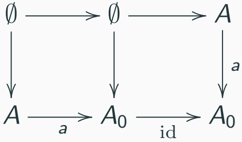
and this is the **strong homotopy cokernel** of $\Gamma(a\colon A \to A_0)$ in $\mathbf{Arr}(\mathcal{A})$

In this Episode

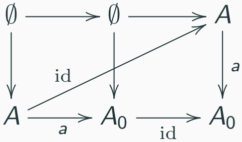
(Danger : spoil !) If the category $\mathcal{A}$ has an initial object $\emptyset$, there is another embedding

$$\Gamma\colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,, \;\; \Gamma(A) = (\emptyset_A\colon \emptyset \to A)$$

For a given arrow $a\colon A \to A_0$ in $\mathcal{A}$, we get the diagram



What is this ? Nothing ! But we get also the diagram



and this is the **strong homotopy cokernel** of $\Gamma(a\colon A \to A_0)$ in $\mathbf{Arr}(\mathcal{A})$

Nullhomotopies

Nullhomotopies

To express the notion of (strong) homotopy cokernel in a category $\mathcal{B}$, we need a **structure Θ of nullhomotopies** on $\mathcal{B}$ (Grandis) :

1. for every arrow $B \xrightarrow{g} C$, a set $\Theta(g)$ of nullhomotopies on g
2. for composable arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, a map

$$f \circ - \circ h \colon \Theta(g) \to \Theta(f \cdot g \cdot h)$$

3. in such a way that, for every $\lambda \in \Theta(g)$,
   - 3.1 $(f' \cdot f) \circ \lambda \circ (g \cdot g') = f' \circ (f \circ \lambda \circ g) \circ g'$
   - 3.2 $\mathrm{id}_B \circ \lambda \circ \mathrm{id}_C = \lambda$

Notation for $\lambda \in \Theta(g)$: $B \overset{g}{\underset{0}{\rightrightarrows}} C$ with $\Uparrow \lambda$

Nullhomotopies

To express the notion of (strong) homotopy cokernel in a category $\mathcal{B}$, we need a **structure Θ of nullhomotopies** on $\mathcal{B}$ (Grandis) :

1. for every arrow $B \xrightarrow{g} C$, a set $\Theta(g)$ of nullhomotopies on g
2. for composable arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, a map

$$f \circ - \circ h \colon \Theta(g) \to \Theta(f \cdot g \cdot h)$$

3. in such a way that, for every $\lambda \in \Theta(g)$,
   3.1 $(f' \cdot f) \circ \lambda \circ (g \cdot g') = f' \circ (f \circ \lambda \circ g) \circ g'$
   3.2 $\mathrm{id}_B \circ \lambda \circ \mathrm{id}_C = \lambda$

Notation for $\lambda \in \Theta(g)$:

Nullhomotopies

To express the notion of (strong) homotopy cokernel in a category $\mathcal{B}$, we need a **structure Θ of nullhomotopies** on $\mathcal{B}$ (Grandis) :

1. for every arrow $B \xrightarrow{g} C$, a set $\Theta(g)$ of nullhomotopies on g
2. for composable arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, a map

$$f \circ - \circ h \colon \Theta(g) \to \Theta(f \cdot g \cdot h)$$

3. in such a way that, for every $\lambda \in \Theta(g)$,
   - 3.1 $(f' \cdot f) \circ \lambda \circ (g \cdot g') = f' \circ (f \circ \lambda \circ g) \circ g'$
   - 3.2 $\mathrm{id}_B \circ \lambda \circ \mathrm{id}_C = \lambda$

Notation for $\lambda \in \Theta(g)$:

Nullhomotopies

To express the notion of (strong) homotopy cokernel in a category $\mathcal{B}$, we need a **structure Θ of nullhomotopies** on $\mathcal{B}$ (Grandis) :

1. for every arrow $B \xrightarrow{g} C$, a set $\Theta(g)$ of nullhomotopies on g
2. for composable arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, a map

$$f \circ - \circ h \colon \Theta(g) \to \Theta(f \cdot g \cdot h)$$

3. in such a way that, for every $\lambda \in \Theta(g)$,
   3.1 $(f' \cdot f) \circ \lambda \circ (g \cdot g') = f' \circ (f \circ \lambda \circ g) \circ g'$
   3.2 $\mathrm{id}_B \circ \lambda \circ \mathrm{id}_C = \lambda$

Nullhomotopies

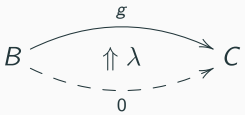
To express the notion of (strong) homotopy cokernel in a category $\mathcal{B}$, we need a **structure Θ of nullhomotopies** on $\mathcal{B}$ (Grandis) :

1. for every arrow $B \xrightarrow{g} C$, a set $\Theta(g)$ of nullhomotopies on g
2. for composable arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, a map

$$f \circ - \circ h \colon \Theta(g) \to \Theta(f \cdot g \cdot h)$$

3. in such a way that, for every $\lambda \in \Theta(g)$,
   3.1 $(f' \cdot f) \circ \lambda \circ (g \cdot g') = f' \circ (f \circ \lambda \circ g) \circ g'$
   3.2 $\mathrm{id}_B \circ \lambda \circ \mathrm{id}_C = \lambda$

Notation for $\lambda \in \Theta(g)$:



Nullhomotopies

Example : if $\mathcal{A} \subseteq \mathcal{B}$ is a subcategory, we put

$$\Theta_{\mathcal{A}}(\, B \xrightarrow{g} C \,) = \{(g_1 \colon B \to A, g_2 \colon A \to C) \mid g_1 \cdot g_2 = g \text{ and } A \in \mathcal{A}\}$$

Example : in particular, if $\mathcal{B}$ has an initial object $\emptyset$, we put

$$\Theta_{\emptyset}(\, B \xrightarrow{g} C \,) = \{\lambda \colon B \to \emptyset \mid \lambda \cdot \emptyset_c = g\}$$

Example : if $\mathcal{B}$ is a 2-category and $\mathcal{Z}$ an ideal of arrows in $\mathcal{B}$, we put

$$\Theta_{\mathcal{Z}}(\, B \xrightarrow{g} C \,) = \left\{ \text{2-cells } \; B \underset{s}{\overset{g}{\Uparrow \lambda}} C \;\middle|\; s \in \mathcal{Z} \right\}$$

Example : in particular, if $\mathcal{B}$ has a zero object 0 and $\mathcal{Z}$ is the ideal of zero arrows, we write Θ_0 instead of $\Theta_{\mathcal{Z}}$. This example justifies the general notation adopted for nullhomotopies.

Nullhomotopies

Example : if $\mathcal{A} \subseteq \mathcal{B}$ is a subcategory, we put

$$\Theta_{\mathcal{A}}(\ B \xrightarrow{g} C\ ) = \{(g_1 \colon B \to A, g_2 \colon A \to C) \mid g_1 \cdot g_2 = g \text{ and } A \in \mathcal{A}\}$$

Example : in particular, if $\mathcal{B}$ has an initial object $\emptyset$, we put

$$\Theta_{\emptyset}(\ B \xrightarrow{g} C\ ) = \{\lambda \colon B \to \emptyset \mid \lambda \cdot \emptyset_c = g\}$$

Example : if $\mathcal{B}$ is a 2-category and $\mathcal{Z}$ an ideal of arrows in $\mathcal{B}$, we put

$$\Theta_{\mathcal{Z}}(\ B \xrightarrow{g} C\ ) = \left\{ \text{2-cells } \ B \underset{s}{\overset{g}{\Uparrow \lambda}} C \ \mid s \in \mathcal{Z} \right\}$$

Example : in particular, if $\mathcal{B}$ has a zero object 0 and $\mathcal{Z}$ is the ideal of zero arrows, we write Θ_0 instead of $\Theta_{\mathcal{Z}}$. This example justifies the general notation adopted for nullhomotopies.

Nullhomotopies

Example : if $\mathcal{A} \subseteq \mathcal{B}$ is a subcategory, we put

$$\Theta_{\mathcal{A}}(\ B \xrightarrow{g} C\ ) = \{(g_1 \colon B \to A, g_2 \colon A \to C) \mid g_1 \cdot g_2 = g \text{ and } A \in \mathcal{A}\}$$

Example : in particular, if $\mathcal{B}$ has an initial object $\emptyset$, we put

$$\Theta_{\emptyset}(\ B \xrightarrow{g} C\ ) = \{\lambda \colon B \to \emptyset \mid \lambda \cdot \emptyset_c = g\}$$

Example : if $\mathcal{B}$ is a 2-category and $\mathcal{Z}$ an ideal of arrows in $\mathcal{B}$, we put

$$\Theta_{\mathcal{Z}}(\ B \xrightarrow{g} C\ ) = \left\{ \text{2-cells } \ B \underset{s}{\overset{g}{\rightrightarrows}} C \ (\Uparrow \lambda) \mid s \in \mathcal{Z} \right\}$$

Example : in particular, if $\mathcal{B}$ has a zero object 0 and $\mathcal{Z}$ is the ideal of zero arrows, we write Θ_0 instead of $\Theta_{\mathcal{Z}}$. This example justifies the general notation adopted for nullhomotopies.

Nullhomotopies

Example : if $\mathcal{A} \subseteq \mathcal{B}$ is a subcategory, we put

$$\Theta_{\mathcal{A}}(\ B \xrightarrow{g} C\ ) = \{(g_1 \colon B \to A, g_2 \colon A \to C) \mid g_1 \cdot g_2 = g \text{ and } A \in \mathcal{A}\}$$

Example : in particular, if $\mathcal{B}$ has an initial object $\emptyset$, we put

$$\Theta_{\emptyset}(\ B \xrightarrow{g} C\ ) = \{\lambda \colon B \to \emptyset \mid \lambda \cdot \emptyset_c = g\}$$

Example : if $\mathcal{B}$ is a 2-category and $\mathcal{Z}$ an ideal of arrows in $\mathcal{B}$, we put

$$\Theta_{\mathcal{Z}}(\ B \xrightarrow{g} C\ ) = \left\{ \text{2-cells } \ B \underset{s}{\overset{g}{\underset{\Uparrow \lambda}{\rightrightarrows}}} C \ \mid s \in \mathcal{Z} \right\}$$

Example : in particular, if $\mathcal{B}$ has a zero object 0 and $\mathcal{Z}$ is the ideal of zero arrows, we write Θ_0 instead of $\Theta_{\mathcal{Z}}$. This example justifies the general notation adopted for nullhomotopies.

Nullhomotopies

Main example : if $\mathcal{A}$ is additive, then **Arr**$(\mathcal{A})$ is a 2-category ; if $\mathcal{A}$ is any category, what remains in **Arr**$(\mathcal{A})$ is a structure of nullhomotopies :

$$\Theta_{\Delta}\left((A, a, A_0) \xrightarrow{(f, f_0)} (B, b, B_0) \right) = \{\lambda \colon A_0 \to B \mid a{\cdot}\lambda = f \text{ and } \lambda{\cdot}b = f_0\}$$

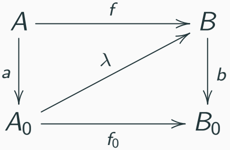
Some more examples are discussed in Episode 2. They are related to :

- generalized (co)radicals and, in particular, (idempotent) (co)monads,
- prepointed categories and, in particular, multipointed categories,
- ideals of arrows seen as discrete structures of nullhomotopies.

Nullhomotopies

Main example : if $\mathcal{A}$ is additive, then $\mathbf{Arr}(\mathcal{A})$ is a 2-category ; if $\mathcal{A}$ is any category, what remains in $\mathbf{Arr}(\mathcal{A})$ is a structure of nullhomotopies :

$$\Theta_{\Delta}\left((A, a, A_0) \xrightarrow{(f, f_0)} (B, b, B_0) \right) = \{\lambda \colon A_0 \to B \mid a \cdot \lambda = f \text{ and } \lambda \cdot b = f_0\}$$



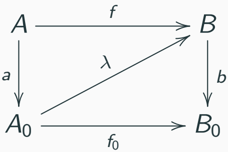
Some more examples are discussed in Episode 2. They are related to :

- generalized (co)radicals and, in particular, (idempotent) (co)monads,
- prepointed categories and, in particular, multipointed categories,
- ideals of arrows seen as discrete structures of nullhomotopies.

Nullhomotopies

Main example : if $\mathcal{A}$ is additive, then $\mathbf{Arr}(\mathcal{A})$ is a 2-category ; if $\mathcal{A}$ is any category, what remains in $\mathbf{Arr}(\mathcal{A})$ is a structure of nullhomotopies :

$$\Theta_\Delta \left((A, a, A_0) \xrightarrow{(f, f_0)} (B, b, B_0) \right) = \{\lambda \colon A_0 \to B \mid a{\cdot}\lambda = f \text{ and } \lambda{\cdot}b = f_0\}$$



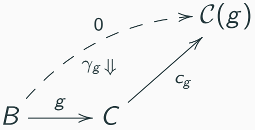
Some more examples are discussed in Episode 2. They are related to :

- generalized (co)radicals and, in particular, (idempotent) (co)monads,
- prepointed categories and, in particular, multipointed categories,
- ideals of arrows seen as discrete structures of nullhomotopies.

Θ-cokernels

Θ-cokernels

In a category with nullhomotopies $(\mathcal{B}, \Theta)$, the Θ-**cokernel** of an arrow $g\colon B \to C$ is a triple $\mathcal{C}(g) \in \mathcal{B}, c_g\colon C \to \mathcal{C}(g), \gamma_g \in \Theta(g \cdot c_g)$

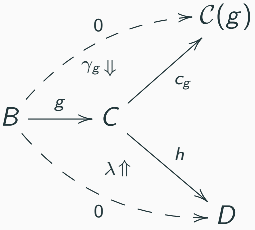


such that, for any other triple $D \in \mathcal{B}, h\colon C \to D, \lambda \in \Theta(g \cdot h)$ there exists a unique arrow $h'\colon \mathcal{C}(g) \to D$ such that

$$c_g \cdot h' = h \quad \text{and} \quad \gamma_g \circ h' = \lambda$$

Θ-cokernels

In a category with nullhomotopies $(\mathcal{B}, \Theta)$, the **Θ-cokernel** of an arrow $g \colon B \to C$ is a triple $\mathcal{C}(g) \in \mathcal{B}, c_g \colon C \to \mathcal{C}(g), \gamma_g \in \Theta(g \cdot c_g)$



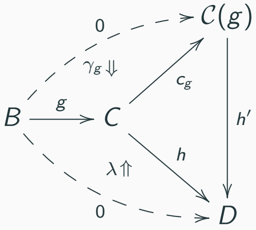
such that, for any other triple $D \in \mathcal{B}, h \colon C \to D, \lambda \in \Theta(g \cdot h)$

there exists a unique arrow $h' \colon \mathcal{C}(g) \to D$ such that

$$c_g \cdot h' = h \;\text{ and }\; \gamma_g \circ h' = \lambda$$

Θ-cokernels

In a category with nullhomotopies $(\mathcal{B}, \Theta)$, the **Θ-cokernel** of an arrow $g\colon B \to C$ is a triple $\mathcal{C}(g) \in \mathcal{B}, c_g\colon C \to \mathcal{C}(g), \gamma_g \in \Theta(g \cdot c_g)$

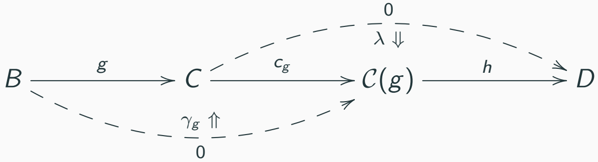


such that, for any other triple $D \in \mathcal{B}, h\colon C \to D, \lambda \in \Theta(g \cdot h)$ there exists a unique arrow $h'\colon \mathcal{C}(g) \to D$ such that

$$c_g \cdot h' = h \ \text{ and } \ \gamma_g \circ h' = \lambda$$

Θ-cokernels

The Θ-cokernel of $g\colon B \to C$ is **strong** when, in the situation

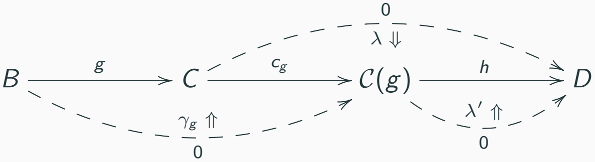


if $\gamma_g \circ h = g \circ \lambda$, then there exists a unique $\lambda' \in \Theta(h)$ such that

$$c_g \circ \lambda' = \lambda$$

Θ-cokernels

The Θ-cokernel of $g\colon B \to C$ is **strong** when, in the situation



if $\gamma_g \circ h = g \circ \lambda$, then there exists a unique $\lambda' \in \Theta(h)$ such that

$$c_g \circ \lambda' = \lambda$$

Θ-cokernels

A couple of egocentric motivations to study Θ-(co)kernels :

1. In Episode 1, with Jacqmin, Mantovani and Metere, we used Θ-(co)kernels to give an internal unified version of the Gabriel-Ulmer exact sequence associated with a functor of pointed groupoids and of the Brown exact sequence associated with a fibration of groupoids.
2. In Episode 2, with Mantovani and Messora, we used Θ-(co)kernels to define a general notion of homotopy torsion theory which includes classical (abelian) torsion theories, torsion theories in multipointed and in prepointed categories, and pretorsion theories.

Θ-cokernels

A couple of egocentric motivations to study Θ-(co)kernels :

1. In Episode 1, with Jacqmin, Mantovani and Metere, we used Θ-(co)kernels to give an internal unified version of the Gabriel-Ulmer exact sequence associated with a functor of pointed groupoids and of the Brown exact sequence associated with a fibration of groupoids.
2. In Episode 2, with Mantovani and Messora, we used Θ-(co)kernels to define a general notion of homotopy torsion theory which includes classical (abelian) torsion theories, torsion theories in multipointed and in prepointed categories, and pretorsion theories.

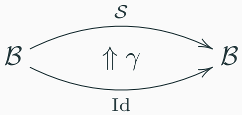
Θ-cokernels

A couple of egocentric motivations to study Θ-(co)kernels :

1. In Episode 1, with Jacqmin, Mantovani and Metere, we used Θ-(co)kernels to give an internal unified version of the Gabriel-Ulmer exact sequence associated with a functor of pointed groupoids and of the Brown exact sequence associated with a fibration of groupoids.
2. In Episode 2, with Mantovani and Messora, we used Θ-(co)kernels to define a general notion of homotopy torsion theory which includes classical (abelian) torsion theories, torsion theories in multipointed and in prepointed categories, and pretorsion theories.

Θ-cokernels

An example from Episode 2 : given a pre-coradical



we get a nullohomotopy structure on $\mathcal{B}$ by

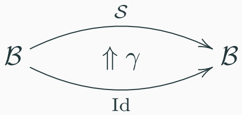
$$\Theta_\gamma(\ B \xrightarrow{g} C\ ) = \{\lambda \colon \mathcal{S}(B) \to C \mid \gamma_B \cdot \lambda = g\}$$

For every object $B \in \mathcal{B}$, the following is the Θ_γ-cokernel of id_B

Moreover, if $\mathcal{S}$ is an idempotent monad and γ is its unit, then this Θ_γ-cokernel is strong

Θ-cokernels

An example from Episode 2 : given a pre-coradical



we get a nullohomotopy structure on $\mathcal{B}$ by

$$\Theta_\gamma(\ B \xrightarrow{g} C\ ) = \{\lambda \colon \mathcal{S}(B) \to C \mid \gamma_B \cdot \lambda = g\}$$

For every object $B \in \mathcal{B}$, the following is the Θ_γ-cokernel of id_B

Moreover, if $\mathcal{S}$ is an idempotent monad and γ is its unit, then this Θ_γ-cokernel is strong

Θ-cokernels

An example from Episode 2 : given a pre-coradical

$$\mathcal{B} \;\underset{\mathrm{Id}}{\overset{\mathcal{S}}{\underset{\Uparrow\, \gamma}{\rightrightarrows}}}\; \mathcal{B}$$

we get a nullohomotopy structure on $\mathcal{B}$ by

$$\Theta_\gamma(\ B \xrightarrow{g} C\ ) = \{\lambda \colon \mathcal{S}(B) \to C \mid \gamma_B \cdot \lambda = g\}$$

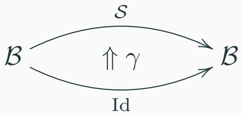
For every object $B \in \mathcal{B}$, the following is the Θ_γ-cokernel of id_B

$$\begin{array}{ccccc} & & \mathcal{S}(B) & & \\ & {\scriptstyle \gamma_B} \nearrow & & \searrow\!\!\searrow {\scriptstyle \mathrm{id}} & \\ B & \xrightarrow[\mathrm{id}]{} & B & \xrightarrow[\gamma_B]{} & \mathcal{S}(B) \end{array}$$

Moreover, if $\mathcal{S}$ is an idempotent monad and γ is its unit, then this Θ_γ-cokernel is strong

Θ-cokernels

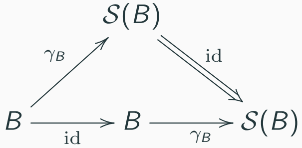
An example from Episode 2 : given a pre-coradical



we get a nullohomotopy structure on $\mathcal{B}$ by

$$\Theta_\gamma(\ B \xrightarrow{g} C\ ) = \{\lambda \colon \mathcal{S}(B) \to C \mid \gamma_B \cdot \lambda = g\}$$

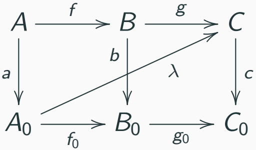
For every object $B \in \mathcal{B}$, the following is the Θ_γ-cokernel of id_B



Moreover, if $\mathcal{S}$ is an idempotent monad and γ is its unit, then this Θ_γ-cokernel is strong

Θ-cokernels

Main example : if $\mathcal{A}$ has pushouts, the Θ_Δ-cokernel in $\mathbf{Arr}(\mathcal{A})$ of an arrow $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ is a diagram of shape

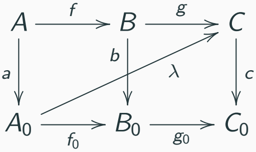


where the two following diagrams should commute

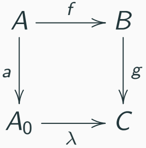
So, just replace these diagrams with the corresponding colimits and we get the strong Θ_Δ-cokernel

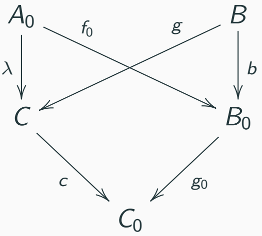
Θ-cokernels

Main example : if $\mathcal{A}$ has pushouts, the Θ_Δ-cokernel in $\mathbf{Arr}(\mathcal{A})$ of an arrow $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ is a diagram of shape



where the two following diagrams should commute

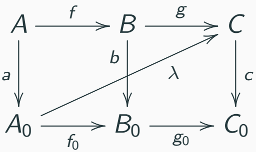




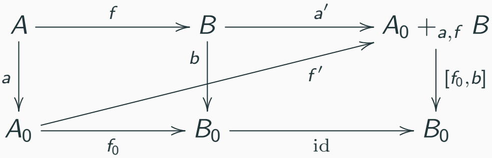
So, just replace these diagrams with the corresponding colimits and we get the strong Θ_Δ-cokernel

Θ-cokernels

Main example : if $\mathcal{A}$ has pushouts, the Θ_Δ-cokernel in $\mathbf{Arr}(\mathcal{A})$ of an arrow $(f, f_0)\colon (A, a, A_0) \to (B, b, B_0)$ is a diagram of shape



So, just replace the two previous diagrams with the corresponding colimits and we get the strong Θ_Δ-cokernel of (f, f_0) in $\mathbf{Arr}(\mathcal{A})$



The heart of the story

The heart of the story

Why, if $\mathcal{A}$ has finite colimits, the embedding

$$\Gamma\colon \mathcal{A} \to \mathbf{Arr}(\mathcal{A})\,, \quad \Gamma(A) = (\emptyset_A\colon \emptyset \to A)$$

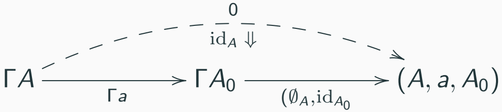
is the completion of $\mathcal{A}$ under strong Θ-cokernels ?

First, the **objects** : we already observed that, for any object $a\colon A \to A_0$ of $\mathbf{Arr}(\mathcal{A})$, the following is a strong Θ_Δ-cokernel

First, the **objects** : we already observed that, for any object $a\colon A \to A_0$ of $\mathbf{Arr}(\mathcal{A})$, the following is a strong Θ_Δ-cokernel

The heart of the story

First, the **objects** : we already observed that, for any object $a \colon A \to A_0$ of $\mathbf{Arr}(\mathcal{A})$, the following is a strong Θ_Δ-cokernel



The heart of the story

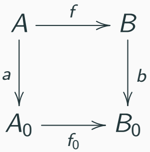
Second, the **arrows** : every arrow in $\mathbf{Arr}(\mathcal{A})$

$$\begin{CD} A @>f>> B \\ @VaVV @VVbV \\ A_0 @>>f_0> B_0 \end{CD}$$

is the unique extension to the Θ_Δ-cokernels

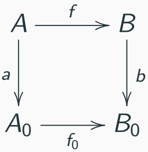
The heart of the story

Second, the **arrows** : every arrow in $\mathbf{Arr}(\mathcal{A})$

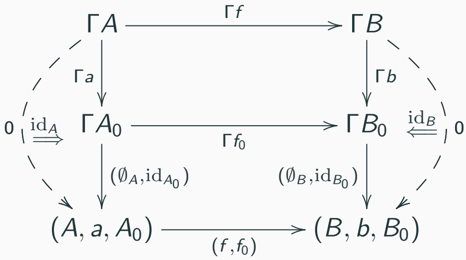


is the unique extension to the Θ_Δ-cokernels

Second, the **arrows** : every arrow in $\mathbf{Arr}(\mathcal{A})$



is the unique extension to the Θ_Δ-cokernels

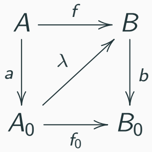


The heart of the story

Third, the **nullhomotopies** : every nullhomotopy in **Arr**($\mathcal{A}$)

is the unique extension to the strong Θ_Δ-cokernels

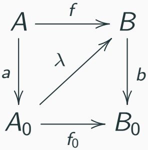
Third, the **nullhomotopies** : every nullhomotopy in $\mathbf{Arr}(\mathcal{A})$



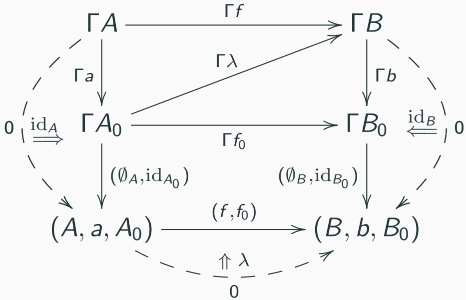
is the unique extension to the strong Θ_Δ-cokernels

The heart of the story

Third, the **nullhomotopies** : every nullhomotopy in **Arr**$(\mathcal{A})$



is the unique extension to the strong Θ_Δ-cokernels



Two technical details

Two technical details

In order to state correctly the main result, we need two technical details.

First : in a category with nullhomotopies $(\mathcal{B}, \Theta)$, the **reduced interchange** (Grandis) is the following condition. In the situation

then $\alpha \circ g = f \circ \beta$

Two technical details

In order to state correctly the main result, we need two technical details.

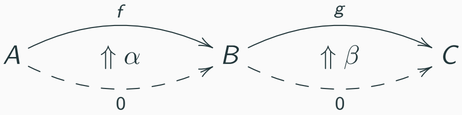
First : in a category with nullhomotopies $(\mathcal{B}, \Theta)$, the **reduced interchange** (Grandis) is the following condition. In the situation

then $\alpha \circ g = f \circ \beta$

Two technical details

In order to state correctly the main result, we need two technical details.

First : in a category with nullhomotopies $(\mathcal{B}, \Theta)$, the **reduced interchange** (Grandis) is the following condition. In the situation



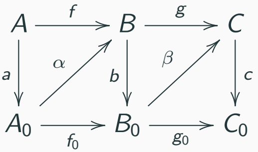
then $\alpha \circ g = f \circ \beta$

Main example : in $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$, the reduced interchange is true

$$\alpha \circ (g, g_0) = \alpha \cdot g = \alpha \cdot b \cdot \beta = f_0 \cdot \beta = (f, f_0) \circ \beta$$

Two technical details

Main example : in $(\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$, the reduced interchange is true



$$\alpha \circ (g, g_0) = \alpha \cdot g = \alpha \cdot b \cdot \beta = f_0 \cdot \beta = (f, f_0) \circ \beta$$

Two technical details

Example :

1. If $\mathcal{B}$ is a 2-category and $\mathcal{Z}$ is an ideal of arrows in $\mathcal{B}$, the reduced interchange in general is not true in $(\mathcal{B}, \Theta_{\mathcal{Z}})$. In fact, the reduced interchange would imply that for every arrow $s \in \mathcal{Z}$ there exists a unique 2-cell $s \Rightarrow s$.
2. Nevertheless, the reduced interchange is true if $\mathcal{B}$ has a 2-zero object and $\mathcal{Z}$ is the ideal of zero arrows.

Two technical details

Example :

1. If $\mathcal{B}$ is a 2-category and $\mathcal{Z}$ is an ideal of arrows in $\mathcal{B}$, the reduced interchange in general is not true in $(\mathcal{B}, \Theta_{\mathcal{Z}})$. In fact, the reduced interchange would imply that for every arrow $s \in \mathcal{Z}$ there exists a unique 2-cell $s \Rightarrow s$.
2. Nevertheless, the reduced interchange is true if $\mathcal{B}$ has a 2-zero object and $\mathcal{Z}$ is the ideal of zero arrows.

Two technical details

Example :

1. If $\mathcal{B}$ is a 2-category and $\mathcal{Z}$ is an ideal of arrows in $\mathcal{B}$, the reduced interchange in general is not true in $(\mathcal{B}, \Theta_{\mathcal{Z}})$. In fact, the reduced interchange would imply that for every arrow $s \in \mathcal{Z}$ there exists a unique 2-cell $s \Rightarrow s$.
2. Nevertheless, the reduced interchange is true if $\mathcal{B}$ has a 2-zero object and $\mathcal{Z}$ is the ideal of zero arrows.

Two technical details

In order to state correctly the main result, we need two technical details.

Second : Θ-**strong colimits** in a category with nullhomotopies $(\mathcal{B}, \Theta)$.

1. An initial object $\emptyset$ is Θ-strong if, for every object $B \in \mathcal{B}$, one has $\Theta(\emptyset_B : \emptyset \to B) = \{*\}$
2. Consider the factorization through the pushout of f and g

The pushout is Θ-strong if, given $\alpha \in \Theta(x)$ and $\beta \in \Theta(y)$ such that $f \circ \alpha = g \circ \beta$, there exists a unique $[\alpha, \beta] \in \Theta([x, y])$ such that $g' \circ [\alpha, \beta] = \alpha$ and $f' \circ [\alpha, \beta] = \beta$

Two technical details

In order to state correctly the main result, we need two technical details.

Second : Θ-**strong colimits** in a category with nullhomotopies $(\mathcal{B}, \Theta)$.

1. An initial object $\emptyset$ is Θ-strong if, for every object $B \in \mathcal{B}$, one has $\Theta(\emptyset_B \colon \emptyset \to B) = \{*\}$
2. Consider the factorization through the pushout of f and g

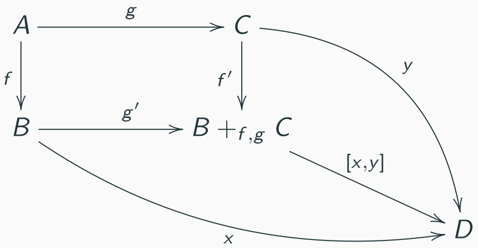
The pushout is Θ-strong if, given $\alpha \in \Theta(x)$ and $\beta \in \Theta(y)$ such that $f \circ \alpha = g \circ \beta$, there exists a unique $[\alpha, \beta] \in \Theta([x, y])$ such that $g' \circ [\alpha, \beta] = \alpha$ and $f' \circ [\alpha, \beta] = \beta$

Two technical details

In order to state correctly the main result, we need two technical details.

Second : Θ-**strong colimits** in a category with nullhomotopies $(\mathcal{B}, \Theta)$.

1. An initial object $\emptyset$ is Θ-strong if, for every object $B \in \mathcal{B}$, one has $\Theta(\emptyset_B \colon \emptyset \to B) = \{*\}$
2. Consider the factorization through the pushout of f and g



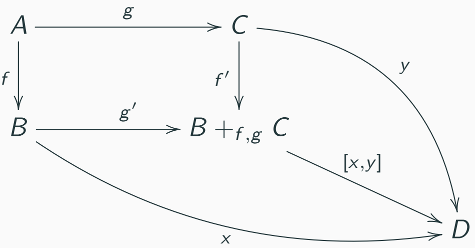
The pushout is Θ-strong if, given $\alpha \in \Theta(x)$ and $\beta \in \Theta(y)$ such that $f \circ \alpha = g \circ \beta$, there exists a unique $[\alpha, \beta] \in \Theta([x, y])$ such that $g' \circ [\alpha, \beta] = \alpha$ and $f' \circ [\alpha, \beta] = \beta$

Two technical details

In order to state correctly the main result, we need two technical details.

Second : Θ-**strong colimits** in a category with nullhomotopies $(\mathcal{B}, \Theta)$.

1. An initial object $\emptyset$ is Θ-strong if, for every object $B \in \mathcal{B}$, one has $\Theta(\emptyset_B : \emptyset \to B) = \{*\}$
2. Consider the factorization through the pushout of f and g



The pushout is Θ-strong if, given $\alpha \in \Theta(x)$ and $\beta \in \Theta(y)$ such that $f \circ \alpha = g \circ \beta$, there exists a unique $[\alpha, \beta] \in \Theta([x, y])$ such that $g' \circ [\alpha, \beta] = \alpha$ and $f' \circ [\alpha, \beta] = \beta$

Two technical details

Main example : if $\mathcal{A}$ has finite colimits, then **Arr**$(\mathcal{A})$ has finite colimits and they are $\Theta_{\mathbf{\Delta}}$-strong

Example : if $\mathcal{B}$ is a 2-category and we take, as ideal of arrows, $\mathcal{Z} = \mathcal{B}$, then to be $\Theta_{\mathcal{Z}}$-strong means to be a 2-colimit

Two technical details

Main example : if $\mathcal{A}$ has finite colimits, then **Arr**$(\mathcal{A})$ has finite colimits and they are Θ_Δ-strong

Example : if $\mathcal{B}$ is a 2-category and we take, as ideal of arrows, $\mathcal{Z} = \mathcal{B}$, then to be $\Theta_{\mathcal{Z}}$-strong means to be a 2-colimit

Two technical details

From Episode 2 : why Θ-strong colimits are useful ?

1. In $(\mathcal{B}, \Theta)$, if $\mathcal{B}$ has strong Θ-cokernels of identity arrows and Θ-strong pushouts, then it has all Θ-cokernels and they are strong.
2. In particular, given a prepointed category

$$\mathcal{A} \; \begin{array}{c} \xleftarrow{\;\mathcal{C}\;} \\ \xrightarrow{\;\mathcal{U}\;} \\ \xleftarrow[\;\mathcal{D}\;]{} \end{array} \; \mathcal{B} \qquad \mathcal{C} \dashv \mathcal{U} \dashv \mathcal{D} \qquad \mathcal{U} \text{ full and faithful}$$

   and the structure Θ on $\mathcal{B}$ induced by the unit of $\mathcal{C} \dashv \mathcal{U}$ or by the counit of $\mathcal{U} \dashv \mathcal{D}$, if $\mathcal{B}$ has pullbacks and pushouts, then they are Θ-strong and $\mathcal{B}$ automatically has strong Θ-kernels and strong Θ-cokernels.

Two technical details

From Episode 2 : why Θ-strong colimits are useful ?

1. In $(\mathcal{B}, \Theta)$, if $\mathcal{B}$ has strong Θ-cokernels of identity arrows and Θ-strong pushouts, then it has all Θ-cokernels and they are strong.
2. In particular, given a prepointed category

$$\mathcal{A} \underset{\mathcal{D}}{\overset{\mathcal{C}}{\leftleftarrows}} \xrightarrow{\mathcal{U}} \mathcal{B} \qquad \mathcal{C} \dashv \mathcal{U} \dashv \mathcal{D} \qquad \mathcal{U} \text{ full and faithful}$$

and the structure Θ on $\mathcal{B}$ induced by the unit of $\mathcal{C} \dashv \mathcal{U}$ or by the counit of $\mathcal{U} \dashv \mathcal{D}$, if $\mathcal{B}$ has pullbacks and pushouts, then they are Θ-strong and $\mathcal{B}$ automatically has strong Θ-kernels and strong Θ-cokernels.

Two technical details

From Episode 2 : why Θ-strong colimits are useful ?

1. In $(\mathcal{B}, \Theta)$, if $\mathcal{B}$ has strong Θ-cokernels of identity arrows and Θ-strong pushouts, then it has all Θ-cokernels and they are strong.
2. In particular, given a prepointed category

$$\mathcal{A} \underset{\mathcal{D}}{\overset{\mathcal{C}}{\leftleftarrows}} \xrightarrow{\mathcal{U}} \mathcal{B} \qquad \mathcal{C} \dashv \mathcal{U} \dashv \mathcal{D} \qquad \mathcal{U} \text{ full and faithful}$$

   and the structure Θ on $\mathcal{B}$ induced by the unit of $\mathcal{C} \dashv \mathcal{U}$ or by the counit of $\mathcal{U} \dashv \mathcal{D}$, if $\mathcal{B}$ has pullbacks and pushouts, then they are Θ-strong and $\mathcal{B}$ automatically has strong Θ-kernels and strong Θ-cokernels.

Two technical details

From Episode 2 : why Θ-strong colimits are useful ?

1. In $(\mathcal{B}, \Theta)$, if $\mathcal{B}$ has strong Θ-cokernels of identity arrows and Θ-strong pushouts, then it has all Θ-cokernels and they are strong.
2. In particular, given a prepointed category

$$\mathcal{A} \underset{\mathcal{D}}{\overset{\mathcal{C}}{\leftleftarrows}} \xrightarrow{\mathcal{U}} \mathcal{B} \qquad \mathcal{C} \dashv \mathcal{U} \dashv \mathcal{D} \qquad \mathcal{U} \text{ full and faithful}$$

   and the structure Θ on $\mathcal{B}$ induced by the unit of $\mathcal{C} \dashv \mathcal{U}$ or by the counit of $\mathcal{U} \dashv \mathcal{D}$, if $\mathcal{B}$ has pullbacks and pushouts, then they are Θ-strong and $\mathcal{B}$ automatically has strong Θ-kernels and strong Θ-cokernels.

The main result

The main result

Main result : If $\mathcal{A}$ is a category with finite colimits, then the embedding

$$\Gamma\colon \mathcal{A} \longrightarrow (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta})$$

is universal among the functors $\mathcal{F}\colon \mathcal{A} \to (\mathcal{B}, \Theta)$ such that

- Θ satisfies the reduced interchange,
- $\mathcal{B}$ has strong Θ-cokernels and Θ-strong finite colimits,
- $\mathcal{F}$ preserves finite colimits

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \Gamma\ } & \mathbf{Arr}(\mathcal{A}) \\ & {\scriptstyle \mathcal{F}} \searrow & \downarrow {\scriptstyle \widehat{\mathcal{F}}} \\ & & \mathcal{B} \end{array}$$

This means that, for any $\mathcal{F}$ as above, there exists an essentially unique morphism of categories with nullhomotopies $\widehat{\mathcal{F}}\colon (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta}) \to (\mathcal{B}, \Theta)$ which preserves Θ-cokernels and finite colimits and such that $\Gamma \cdot \widehat{\mathcal{F}} \simeq \mathcal{F}$

The main result

Main result : If $\mathcal{A}$ is a category with finite colimits, then the embedding

$$\Gamma\colon \mathcal{A} \longrightarrow (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta})$$

is universal among the functors $\mathcal{F}\colon \mathcal{A} \to (\mathcal{B}, \Theta)$ such that

- Θ satisfies the reduced interchange,
- $\mathcal{B}$ has strong Θ-cokernels and Θ-strong finite colimits,
- $\mathcal{F}$ preserves finite colimits

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma} & \mathbf{Arr}(\mathcal{A}) \\ & {\scriptstyle \mathcal{F}} \searrow & \downarrow {\scriptstyle \widehat{\mathcal{F}}} \\ & & \mathcal{B} \end{array}$$

This means that, for any $\mathcal{F}$ as above, there exists an essentially unique morphism of categories with nullhomotopies $\widehat{\mathcal{F}}\colon (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta}) \to (\mathcal{B}, \Theta)$ which preserves Θ-cokernels and finite colimits and such that $\Gamma \cdot \widehat{\mathcal{F}} \simeq \mathcal{F}$

The main result

Main result : If $\mathcal{A}$ is a category with finite colimits, then the embedding

$$\Gamma\colon \mathcal{A} \longrightarrow (\mathbf{Arr}(\mathcal{A}), \Theta_{\Delta})$$

is universal among the functors $\mathcal{F}\colon \mathcal{A} \to (\mathcal{B}, \Theta)$ such that

- Θ satisfies the reduced interchange,
- $\mathcal{B}$ has strong Θ-cokernels and Θ-strong finite colimits,

The main result

Main result : If $\mathcal{A}$ is a category with finite colimits, then the embedding

$$\Gamma\colon \mathcal{A} \longrightarrow (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$$

is universal among the functors $\mathcal{F}\colon \mathcal{A} \to (\mathcal{B}, \Theta)$ such that

- Θ satisfies the reduced interchange,
- $\mathcal{B}$ has strong Θ-cokernels and Θ-strong finite colimits,
- $\mathcal{F}$ preserves finite colimits

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma} & \mathbf{Arr}(\mathcal{A}) \\ & {\scriptstyle \mathcal{F}} \searrow & \downarrow {\scriptstyle \widehat{\mathcal{F}}} \\ & & \mathcal{B} \end{array}$$

This means that, for any $\mathcal{F}$ as above, there exists an essentially unique morphism of categories with nullhomotopies $\widehat{\mathcal{F}}\colon (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta) \to (\mathcal{B}, \Theta)$ which preserves Θ-cokernels and finite colimits and such that $\Gamma \cdot \widehat{\mathcal{F}} \simeq \mathcal{F}$

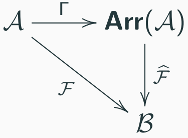
The main result

Main result : If $\mathcal{A}$ is a category with finite colimits, then the embedding

$$\Gamma\colon \mathcal{A} \longrightarrow (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$$

is universal among the functors $\mathcal{F}\colon \mathcal{A} \to (\mathcal{B}, \Theta)$ such that

- Θ satisfies the reduced interchange,
- $\mathcal{B}$ has strong Θ-cokernels and Θ-strong finite colimits,
- $\mathcal{F}$ preserves finite colimits



This means that, for any $\mathcal{F}$ as above, there exists an essentially unique morphism of categories with nullhomotopies $\widehat{\mathcal{F}}\colon (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta) \to (\mathcal{B}, \Theta)$ which preserves Θ-cokernels and finite colimits and such that $\Gamma \cdot \widehat{\mathcal{F}} \simeq \mathcal{F}$

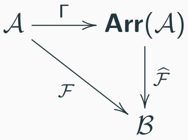
The main result

Main result : If $\mathcal{A}$ is a category with finite colimits, then the embedding

$$\Gamma \colon \mathcal{A} \longrightarrow (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta)$$

is universal among the functors $\mathcal{F} \colon \mathcal{A} \to (\mathcal{B}, \Theta)$ such that

- Θ satisfies the reduced interchange,
- $\mathcal{B}$ has strong Θ-cokernels and Θ-strong finite colimits,
- $\mathcal{F}$ preserves finite colimits



This means that, for any $\mathcal{F}$ as above, there exists an essentially unique morphism of categories with nullhomotopies $\widehat{\mathcal{F}} \colon (\mathbf{Arr}(\mathcal{A}), \Theta_\Delta) \to (\mathcal{B}, \Theta)$ which preserves Θ-cokernels and finite colimits and such that $\Gamma \cdot \widehat{\mathcal{F}} \simeq \mathcal{F}$

Forthcoming Episode

Forthcoming Episode

In the forthcoming Episode 4, with Dupont we will show that, if $\mathcal{A}$ is an abelian category, then the bicategory of fractions

$$\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$$

is a 2-abelian bicategory.

Here Σ is a class of arrows obtained by intersection from two factorization systems constructed in $\mathbf{Arr}(\mathcal{A})$ using Θ_Δ-kernels and Θ_Δ-cokernels.

See you in Palermo for PSSL 108 in September !

Forthcoming Episode

In the forthcoming Episode 4, with Dupont we will show that, if $\mathcal{A}$ is an abelian category, then the bicategory of fractions

$$\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$$

is a 2-abelian bicategory.

Here Σ is a class of arrows obtained by intersection from two factorization systems constructed in $\mathbf{Arr}(\mathcal{A})$ using Θ_Δ-kernels and Θ_Δ-cokernels.

See you in Palermo for PSSL 108 in September !

Forthcoming Episode

In the forthcoming Episode 4, with Dupont we will show that, if $\mathcal{A}$ is an abelian category, then the bicategory of fractions

$$\mathbf{Arr}(\mathcal{A})[\Sigma^{-1}]$$

is a 2-abelian bicategory.

Here Σ is a class of arrows obtained by intersection from two factorization systems constructed in $\mathbf{Arr}(\mathcal{A})$ using Θ_Δ-kernels and Θ_Δ-cokernels.

See you in Palermo for PSSL 108 in September !