BICATEGORICAL PRESENTATIONS OF ÉTENDUES

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Étendues

Definition A Grothendieck topos $\mathcal{A}$ is called an *étendue* if there is an object $A \in \mathcal{A}$ such that

- $A \longrightarrow 1$ is epic
- the slice topos $\mathcal{A}/A$ is localic

Slogan Étendues are the Grothendieck topoi that locally look like a locale

Examples $\mathbf{Set}^G$, $\mathbf{Set}^\Omega$, $\mathbf{Top}(X, G)$, $\mathbf{Set}^{\mathcal{C}^{op}}$ where all morphisms in $\mathcal{C}$ are monic maps

G $\quad (\mathbb{N}, s)$ $\quad G \times 1$ $\quad \coprod_{c \in \mathcal{C}} C$

Presentations of étendues: localic groupoids

ON OBJECTS

The classifying topos of an étale localic groupoid is an étendue [SGA4, Joyal-Tierney]. Any étendue can be recovered as the classifying topos of an étale localic groupoid

1-CATEGORICALLY

[Moerdijk] There is a functor B [Étale groupoids] $\longrightarrow$ [Étendues] inducing an equivalence [Étale groupoids][W^{-1}] $\xrightarrow{\simeq}$ [Étendues]

essential equivalences

BICATEGORICALLY (for spatial étendues)

[Pronk] There is a bifunctor B [Étale groupoids] $\longrightarrow$ [Étendues] inducing a biequivalence [T1-Étale groupoids][W^{-1}] $\xrightarrow{\simeq}$ [T1-Étendues]

essential equivalences

Presentations of étendues: sites with only monics

Definition A small category is called **left cancellative** if all its morphisms are monic

Example A left cancellative category with one object is a left cancellative monoid

$\boxed{\text{ON OBJECTS}}$

[Rosenthal] The topos of sheaves on a left cancellative Grothendieck site is an étendue

[Kock-Moerdijk] Any étendue can be recovered as a topos of sheaves on a left cancellative site

Presentations of étendues Ehresmann sites I

Definition An ordered groupoid is an internal groupoid G in the category of posets with domain $d : G_1 \to G_0$ a fibration

[DeWolf-Pronk] We can interpret G as a double category

$$\begin{tikzcd} A \arrow[r, "f", "\cong"'] \arrow[d, no head] \arrow[dr, phantom, "\geq"] & B \arrow[d, no head] \\ A' \arrow[r, "\cong", "f'"'] & B' \end{tikzcd}$$

order in G_1 — objects in G_0 — partial order in G_0 — object in G_1

$+$ any $\begin{tikzcd} A' \arrow[d] & \\ A \arrow[r, "f", "\cong"'] & B \end{tikzcd}$ $\exists!$ $\begin{tikzcd} A' \arrow[r, dashed, "f|_{A'}"] \arrow[d] \arrow[dr, phantom, "\geq"] & B' \arrow[d, dashed] \\ A \arrow[r, "f", "\cong"'] & B \end{tikzcd}$

Definition An Ehresmann topology E on an ordered groupoid G is an assignment for all $A \in G$ of a collection $E(A)$ of vertical sieves on A (i.e. downclosed subsets of $\downarrow A$) subject to

(E1) $\downarrow A \in E(A)$

(E2) For all $S \in E(A)$ and all $\begin{tikzcd} B \arrow[r, "f"] & A' \arrow[d] \\ & A \end{tikzcd}$, the vertical ideal $f^* S \in E(B)$

(E3) Let $S \in E(A)$ and $R \subseteq \downarrow A$. If for all $\begin{tikzcd} B \arrow[r, "f"] & A' \arrow[d, "\text{in } S"] \\ & A \end{tikzcd}$, $f^* R \in E(B) \Rightarrow R \in E(A)$

Presentations of étendues Ehresmann sites II

Definition An Ehresmann site is an ordered groupoid endowed with an Ehresmann topology.

Definition A presheaf on an Ehresmann site (G,E) is a double functor $F\ E^{op,op} \longrightarrow QSet$. A sheaf on (G,E) is a presheaf st for all $A \in G$, $S \in E(A)$ and "compatible family" $\{s_1 \in F(A_1)\}_{A_1 \leq A \text{ in } S}$ there exists a unique $s \in F(A)$ glueing the family

ON OBJECTS

[Lawson-Steinberg] The category of sheaves on an Ehresmann site is an étendue

[DeWolf-Pronk] Any étendue is the category of sheaves on an Ehresmann site

Left cancellative sites & Ehresmann sites I

1-CATEGORICALLY

each $A \in G_0$ is $\leq$ than a unique maximal element

$$\begin{tikzcd}
{[\text{Ordered groupoids}]_{max}} \arrow[d, shift left, "C"] \arrow[rrr, squiggly, "+\text{topology}"'] & & & {[\text{Ehresmann sites}]_{max}} \arrow[d, "C"] \arrow[dr, "sh(-)"] \arrow[d, phantom, shift left=6, "\circlearrowleft"] & \\
{[\text{Left cancellative cats}]} \arrow[u, shift left, "G"] & & & {[\text{Left cancellative sites}]} \arrow[r, "sh(-)"'] & {[\text{Étendues}]}
\end{tikzcd}$$

[Lawson]

[Lawson-Steinberg]

BICATEGORICALLY

$$\begin{tikzcd}
{[\text{Ordered groupoids}]_{max}} \arrow[d, shift left, "C"] \arrow[d, phantom, "\simeq"] \arrow[rrr, squiggly, "+\text{topology}"'] & & & {[\text{Ehresmann sites}]_{max}} \arrow[d, shift left, "C"] \arrow[d, phantom, "\simeq"] \arrow[dr, "sh(-)"] \arrow[d, phantom, shift left=6, "\circlearrowleft"] & \\
{[\text{Left cancellative cats}]} \arrow[u, shift left, "G"] & & & {[\text{Left cancellative sites}]} \arrow[u, shift left, "G"] \arrow[r, "sh(-)"'] & {[\text{Étendues}]}
\end{tikzcd}$$

biequivalence

[DeWolf-Pronk]

Left cancellative sites & Ehresmann sites II

$$\begin{tikzcd}
{[\text{Ehresmann sites}]_{max}} \arrow[rr, "Sh_E(-)"] \arrow[d, leftrightarrow, "\simeq"] & & {[\text{Étendues}]} \\
{[\text{Left cancellative sites}]} \arrow[urr, "Sh_G(-)"'] & &
\end{tikzcd}
\quad \overset{?}{\rightsquigarrow} \quad
\begin{tikzcd}
{[\text{Ehresmann sites}]_{max}[W_E^{-1}]} \arrow[rr, "Sh_E(-)", "\simeq"'] \arrow[d, leftrightarrow, "\simeq"] & & {[\text{Étendues}]} \\
{[\text{Left cancellative sites}][W_G^{-1}]} \arrow[urr, "Sh_G(-)"', "\simeq"] & &
\end{tikzcd}$$

Using $[\text{Grothendieck sites}][LC^{-1}] \xrightarrow[\simeq]{Sh_G(-)} [\text{Grothendieck topoi}]$, we have that our most natural candidate for W_G is $LC|_{\text{left canc}}$

However, the cospan representing a geometric morphism between étendues may not have a left cancellative vertex!

$F\ \underbrace{Sh_G(A)}_{\text{left canc}} \to \underbrace{Sh_G(B)}_{\text{left canc}}$ $\rightsquigarrow$ $A \rightarrow e \leftarrow B$ s.t. $\begin{tikzcd} Sh_G(A) \arrow[r] \arrow[rr, bend left, "F"] & Sh_G(e) \arrow[r, "\simeq"] & Sh_G(B) \end{tikzcd}$ but e not left cancellative

Enlarging [left cancellative sites] Step 1

We transport the notion of locally monic map for topoi [Kock-Moerdijk] to Grothendieck sites

Definition Let $(\mathcal{C}, T)$ be a Grothendieck site. We say that a morphism $f\ A \to B$ in $\mathcal{C}$ is locally monic if there exist a covering $(g_i\ A_i \to A)_{i \in I}$ such that fg_i is a monomorphism $\forall i \in I$

Remark Monic maps are locally monic

Proposition Let $(\mathcal{C}, T)$ be a site in which each morphism is locally monic. Then $Sh(\mathcal{C}, T)$ is an étendue

$$\begin{tikzcd}
\text{[left cancellative site]} \arrow[r, hook] \arrow[rr, bend left, "Sh(-)"] & \text{[locally monic site]} \arrow[r, "Sh(-)"] & \text{[Étendue]}
\end{tikzcd}$$

Enlarging [left cancellative sites] Step 2

We transport the notion of torsion-free object for topoi [Kock-Moerdijk] to Grothendieck sites

Definition Let $(\mathcal{C}, T)$ be a Grothendieck site. We say that an object $X \in \mathcal{C}$ is torsion-free if every morphism with codomain X is locally monic. We say $(\mathcal{C}, T)$ is generated by torsion-free objects if every object has a covering by torsion-free objects.

Remark If all morphisms in a site are locally monic, then the site is generated by torsion-free objects.

> **Proposition** If $(\mathcal{C}, T)$ is generated by torsion-free objects, then $\mathrm{Sh}(\mathcal{C}, T)$ is an étendue

$$\begin{tikzcd}
{[\mathrm{LCsite}]} \arrow[r, hook] \arrow[rrr, bend left, "\mathrm{Sh}(-)"] & {[\mathrm{LMSite}]} \arrow[r, hook] & {[\mathrm{TFGSite}]} \arrow[r, "\mathrm{Sh}(-)"] & {[\text{Étendue}]}
\end{tikzcd}$$

TFG sites Bicategory of fractions

Definition A morphism of Grothendieck sites $f\ (\mathcal{A}, T_{\mathcal{A}}) \longrightarrow (\mathcal{B}, T_{\mathcal{B}})$ (i.e. covering-preserving & covering-flat functor) is an LC-morphism if

- f is "essentially surjective up to coverings in $(\mathcal{B}, T_{\mathcal{B}})$"
- f is "fully-faithful up to coverings in $(\mathcal{A}, T_{\mathcal{A}})$"

from Lemme de Comparaison

Proposition The class of LC-morphisms in [TFGSite] admits a left calculus of fractions

Theorem The pseudofunctor Sh(-) $[\text{TFGsite}] \longrightarrow [\text{Étendue}]$ sends LC-morphisms to equivalences and induces a biequivalence

$$\mathrm{Sh}(-)\ [\text{TFGsite}][LC^{-1}] \xrightarrow{\simeq} [\text{Étendue}]$$

Enlarging [Ehresmann site] Step 1

ORDERED GROUPOID G

$$\begin{tikzcd} A \arrow[r, "f", "\cong"'] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "\cong", "f'"'] & B' \end{tikzcd}$$

order in G_1

objects in G_0

partial order in G_0

object in G_1

groupoid internal to Poset

+

any
$$\begin{tikzcd} A' \arrow[d] & \\ A \arrow[r, "f", "\cong"'] & B \end{tikzcd}$$
$\exists !$
$$\begin{tikzcd} A' \arrow[r, dashed, "f|_{A'}"] \arrow[d] \arrow[dr, phantom, "\geq"] & B \arrow[d, dashed] \\ A \arrow[r, "f", "\cong"'] & B \end{tikzcd}$$

domain is a fibration

ORDERED CATEGORY P

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

order in P_1

objects in P_0

partial order in P_0

object in P_1

category internal to Poset

+

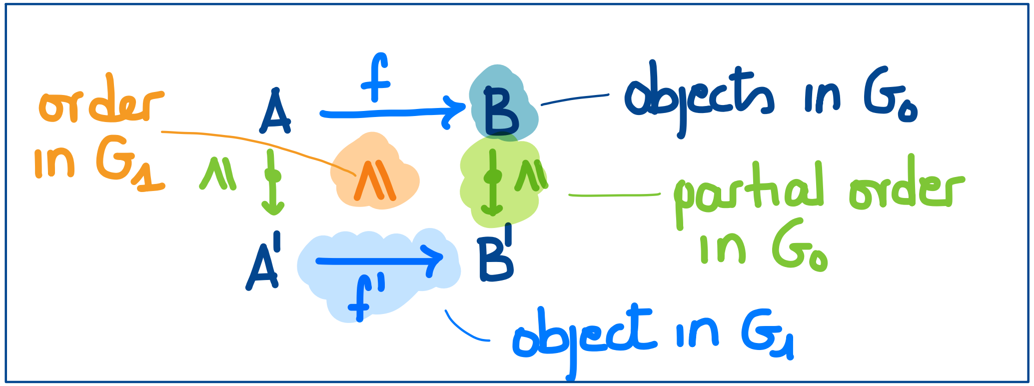
any
$$\begin{tikzcd} A' \arrow[d] & \\ A \arrow[r, "f"] & B \end{tikzcd}$$
$\exists !$
$$\begin{tikzcd} A' \arrow[r, dashed, "f|_{A'}"] \arrow[d] \arrow[dr, phantom, "\geq"] & B \arrow[d, dashed] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

domain is a discrete fibration

TWO DIFFERENCES (1) Horizontal arrows are not isos (2) discrete fibration

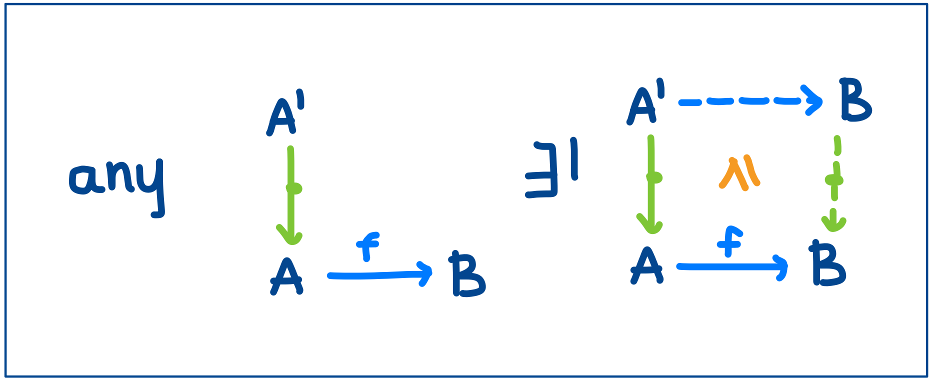
Enlarging [Ehresmann site] Step 2

ORDERED CATEGORY P



category internal to Poset

+



domain is a discrete fibration

TOPOLOGICAL INFORMATION

VERTICAL DIRECTION

Ehresmann topology

HORIZONTAL DIRECTION

Idea We want a single horizontal arrow to be "covering" and this compatibly with the vertical topology

Enlarging [Ehresmann site] Step 3

HORIZONTAL TOPOLOGICAL INFORMATION

$\boxed{HQ}$ For any diagram
$$\begin{tikzcd} & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$
$f, g \in P_1$

there exist $f', g' \in P_1$ st
$$\begin{tikzcd} \cdot \arrow[r, dashed, "f'"] \arrow[d, dashed, "g'"'] \arrow[dr, phantom, "\circlearrowleft"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

$\boxed{RE}$ Every $\xrightarrow{f}$ in P_1 is a horizontal regular epi

"Horizontal atomic coverage"

$\boxed{VQ}$ For any diagram
$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$
$f \in P_1$

there exists $f' \in P_1$ and a 2-cell
$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\Uparrow"] & \cdot \arrow[d] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

compatibility with the vertical direction

Definition A generalized Ehresmann site = P ordered category + vertical Ehresmann topology J + $\boxed{HQ}$ + $\boxed{RE}$ + $\boxed{VQ}$

Enlarging [Ehresmann site] Step 4

ÉTENDUE-LIKE BEHAVIOUR

Definition A horizontal arrow $A \xrightarrow{f} B$ in P is locally monic if there exists a vertical covering family $(A_i \twoheadrightarrow A)_{i \in I}$ s.t. for all $i \in I$ $f|_{A_i}$ is a (horizontal) isomorphism

Definition An object X in P is torsion-free if every horizontal morphism $A \xrightarrow{f} X$ is locally monic

Definition A generalized Ehresmann site (P, J) is torsion-free generated if each object $A \in P$ has a vertical covering $(A_i \twoheadrightarrow A)_{i \in I}$ in the topology T such that $\forall i \in I$ there is a horizontal arrow $B_i \xrightarrow{f_i} A_i$ with B_i torsion-free object

Sheaves on TFG generalized Ehresmann sites

Definition

- A presheaf on a TFG generalized Ehresmann site (P,J) is a double functor $F\ P^{op,op} \longrightarrow QSet$

- A presheaf F on (P,J) is called a sheaf if

  (1) it is a sheaf for the Ehresmann topology J

  (2) for all $A \xrightarrow{f} B$ horizontal arrow and all $x \in F(A)$ self-compatible element [i.e. for all pairs $\begin{array}{ccc} X & \xrightarrow{g} & X_1 \\ & & \updownarrow \\ & & A \end{array}$, $\begin{array}{ccc} X & \xrightarrow{h} & X_2 \\ & & \updownarrow \\ & & A \end{array}$ s.t. $f|_{X_1} g = f|_{X_2} h$ we have $F\left(\begin{array}{ccc} X & \xrightarrow{g} & X_1 \\ & & \updownarrow \\ & & A \end{array}\right)(x) = F\left(\begin{array}{ccc} X & \xrightarrow{h} & X_2 \\ & & \updownarrow \\ & & A \end{array}\right)(x)$], there exists a unique $y \in F(B)$ with $F(f)(y) = x$

TFG generalized Ehresmann & TFG Grothendieck sites 1

Proposition To any TFG generalized Ehresmann site (P, J) we can associate a TFG Grothendieck site $G(P,J)$ such that

$$Sh_E(P,J) \cong Sh_G(G(P,J))$$

Description of $G(P,J)$

OBJECTS	MORPHISMS	TOPOLOGY
the objects of P	formal compositions $A \xrightarrow{f} A' \rightarrow B$	$(A_i \xrightarrow{f_i} A_i' \rightarrow A)_{i \in I}$ covers iff $(A_i' \rightarrow A)_{i \in I}$ covers in (P,J)

Corollary If (P,J) is a TFG generalized Ehresmann site then $Sh_E(P,J)$ is an étendue

TFG generalized Ehresmann & TFG Grothendieck sites 2

Proposition To any TFG Grothendieck site (G,T) we can associate a TFG generalized Ehresmann site $E(G,T)$ such that $Sh_G(G,T) \simeq Sh_E(E(G,T))$

Construction of $E(G,T)$

1

Find $(G,T) \to (\bar{G},\bar{T})$ LC-morphism

- TFG
- finitely complete
- reg epi-mono stable orthogonal FS

2

OBJECTS

Subobjects of $\bar{G}$

$[m \; A \hookrightarrow B]$

2-CELLS

$h \; A \twoheadrightarrow C$

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & A' \arrow[d] \\ & B \end{tikzcd} \qquad \begin{tikzcd} {[m \; A \hookrightarrow B]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong \text{ comm}"] & {[n \; C \hookrightarrow D]} \arrow[d] \\ {[m' \; A' \hookrightarrow B]} \arrow[r] & {[n' \; C' \hookrightarrow D]} \end{tikzcd}$$

TOPOLOGY $([A_i \hookrightarrow A] \to [A' \hookrightarrow A])$ covering iff $(A_i \hookrightarrow A')_{i \in I}$ covering for $\bar{T}$

Corollary Any étendue can be recovered as a category of sheaves on a TFG generalized Ehresmann site

Presentations of étendues TFG generalized Ehresmann sites

ON OBJECTS

Theorem If (P,J) is a TFG generalized Ehresmann site then $Sh_E(P,J)$ is an étendue

Theorem Any étendue can be recovered as a category of sheaves on a TFG generalized Ehresmann site

BICATEGORICALLY ??? work currently in progress

We know $\begin{bmatrix}\text{TFG Grothendieck + pullbacks} \\ \text{+ RE-M stable orthogonal FS}\end{bmatrix}[LC^{-1}] \simeq [\text{Étendues}]$

We claim $\begin{bmatrix}\text{TFG Grothendieck + pullbacks} \\ \text{+ RE-M stable orthogonal FS}\end{bmatrix} \simeq \begin{bmatrix}\text{TFG generalized Ehresmann} \\ \text{+ max + "pullbacks"}\end{bmatrix}$

Thank you for
your attention

References

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[DeWolf-Pronk] A double categorical view on representations of étendues

[Joyal-Tierney] An extension of the Galois theory of Grothendieck

[Kock-Moerdijk] Presentations of étendues

[Lawson] Ordered groupoids and left cancellative categories

[Lawson-Steinberg] Ordered groupoids and étendues

[Moerdijk] The classifying topos of a continuous groupoid I

[Pronk] Etendues and stacks as bicategories of fractions

[Rosenthal] Étendues and categories with monic maps