Protoadditive Functors and Pretorsion Theories in a Multipointed Context

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Framework and Preliminaries

Assumptions

We assume to work in a Barr-exact and protomodular category $\mathbb{C}$ with finite coproducts. We define $\mathcal{Z}$ as the full replete subcategory whose objects are (modulo isomorphisms) the initial object $\mathbf{2}$ and the terminal object $\mathbf{1}$. We denote the class of arrows of $\mathbb{C}$ factorizing through an object of $\mathcal{Z}$ as $N_{\mathcal{Z}}$.

Moreover, we require:

a) $\mathbf{2} \neq \mathbf{1}$;

b) for every object $A \neq \mathbf{1}$, the unique arrow $\iota_A \colon \mathbf{2} \to A$ is a monomorphism;

c) for every object A, if there exists an arrow $e \colon \mathbf{1} \to A$ then $A = \mathbf{1}$ and $e = id_{\mathbf{1}}$.

Examples

Examples

Categories satisfying these assumptions include the category $\mathbb{B}$oole of Boolean algebras, the category $\mathbb{MV}$ of MV-algebras, the category $\mathbb{H}$eyt of Heyting algebras, and the category $\mathbb{S}\text{et}^{\text{op}}$.

For an elementary topos $\mathbb{E}$. TFAE:

i) for every object $A \neq \mathbf{2}$, the unique arrow $\tau_A \colon A \to \mathbf{1}$ is an epimorphism;

ii) $\mathbb{E}$ is two-valued (i.e. $\mathrm{Sub}(\mathbf{1})$ has exactly two elements).

Example

Every category of the form $\mathbb{E}^{op}$, where $\mathbb{E}$ is a two-valued elementary topos.

$\mathcal{Z}$-prekernels and $\mathcal{Z}$-precokernels

Definition (Facchini, Finocchiaro[1])

Let $f\colon A \to B$ be a morphism in $\mathbb{C}$. We say that a morphism $k\colon K \to A$ in $\mathbb{C}$ is a *$\mathcal{Z}$-prekernel* of f if:

- $fk \in N_{\mathcal{Z}}$;
- if $fe \in N_{\mathcal{Z}}$, then there exists a unique morphism φ in $\mathbb{C}$ such that $k\varphi = e$.

In our case, the prekernel of $f\colon A \to B$, if $B \neq \mathbf{1}$, is given by:

$$\begin{array}{ccc} K[f] & \xrightarrow{\;g\;} & \mathbf{2} \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle \iota_B} \\ A & \xrightarrow[\;f\;]{} & B. \end{array}$$

The prekernel of $\tau_A\colon A \to \mathbf{1}$ is id_A.

[1] *Pretorsion theories, stable category and preordered sets*, Annali di Matematica Pura ed Applicata.

$\mathcal{Z}$-prekernels and $\mathcal{Z}$-precokernels

Dually, we have the definition of $\mathcal{Z}$-precokernel.

Not every arrow admits a $\mathcal{Z}$-precokernel: if the two projections $\pi_1, \pi_2 \colon \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ are not equal (or, equivalently, $\mathbf{2} \times \mathbf{2}$ is not isomorphic to $\mathbf{2}$) the arrow $id_{\mathbf{2} \times \mathbf{2}} \colon \mathbf{2} \times \mathbf{2} \to \mathbf{2} \times \mathbf{2}$ does not admit a $\mathcal{Z}$-precokernel.

If the $\mathcal{Z}$-precokernel of f exists, then it is given by a pushout of the form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \chi}\downarrow & \ulcorner & \downarrow{\scriptstyle q} \\ Z & \longrightarrow & Q, \end{array}$$

where Z is an object of $\mathcal{Z}$ and it depends on f.

Pretorsion Theories

Consider $f\colon A \to B$ and $g\colon B \to C$ in $\mathbb{C}$. We say that

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a *short $\mathcal{Z}$-pre-exact sequence* in $\mathbb{C}$ if f is a $\mathcal{Z}$-prekernel of g and g is a $\mathcal{Z}$-precokernel of f.

Definition (Facchini, Finocchiaro, Gran[2])

A *pretorsion theory* $(\mathcal{T}, \mathcal{F})$ in a category $\mathbb{C}$ consists of two full, replete subcategories $\mathcal{T}, \mathcal{F}$ of $\mathbb{C}$ satisfying the following two conditions. Set $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$:

- $\mathrm{Hom}_{\mathbb{C}}(T, F) \subseteq N_{\mathcal{Z}}$ for every object $T \in \mathcal{T}$, $F \in \mathcal{F}$;
- for every object A of $\mathbb{C}$ there is a short $\mathcal{Z}$-pre-exact sequence

$$T(A) \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} F(A)$$

  with $T(A) \in \mathcal{T}$ and $F(A) \in \mathcal{F}$. This $\mathcal{Z}$-pre-exact sequence is unique up to isomorphisms.

[2] *Pretorsion theories in general categories*, Journal of Pure and Applied Algebra.

Pretorsion Theories

Every pretorsion theory defines two functors:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathcal{F} \\ A & \longmapsto & F(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ B & \longmapsto & F(B) \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{T} & \mathcal{T} \\ A & \longmapsto & T(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle T(f)} \\ B & \longmapsto & T(B) \end{array}$$

where $T(f)$ and $F(f)$ are determined by the universal properties of $\mathcal{Z}$-prekernel and $\mathcal{Z}$-precokernel.
Moreover:

- the functor $F\colon \mathbb{C} \to \mathcal{F}$ is a left inverse left adjoint of the inclusion functor $i_{\mathcal{F}}\colon \mathcal{F} \hookrightarrow \mathbb{C}$ and the unit is given by η;
- the functor $T\colon \mathbb{C} \to \mathcal{T}$ is a left inverse right adjoint of the inclusion functor $i_{\mathcal{T}}\colon \mathcal{T} \hookrightarrow \mathbb{C}$ and the counit is given by ε.

Admissible Galois Structures

Definition (Janelidze[3])

A *Galois structure* consists in an adjunction $S \dashv C \colon \mathbb{P} \to \mathbb{A}$ and two admissible classes $\mathcal{A} \subseteq \mathbb{A}$, $\mathcal{P} \subseteq \mathbb{P}$ of arrows, such that

- $S(\mathcal{A}) \subseteq \mathcal{P}$ and $C(\mathcal{P}) \subseteq \mathcal{A}$,
- for every object A of $\mathbb{A}$ the component η_A of the unit of the adjunction $S \dashv C$ is in $\mathcal{A}$ and for every object X of $\mathbb{P}$ the component ε_X of the counit of the adjunction $S \dashv C$ is in $\mathcal{P}$.

We define the functors: $S_A \colon \mathcal{A}/A \to \mathcal{P}/S(A)$ and $C_A \colon \mathcal{P}/S(A) \to \mathcal{A}/A$. One has $S_A(f \colon B \to A) = S(f)$ and $C_A(g \colon P \to S(A)) = \pi_A$ where π_A is the pullback of $C(g)$ along η_A. S_A is the left adjoint of C_A.

Definition (Janelidze[3])

A Galois structure $(S, C, \mathcal{A}, \mathcal{P})$ is *admissible* when the functor C_A is full and faithful for every object A of $\mathbb{A}$.

[3] *Pure Galois theory in categories*, Journal of Algebra.

Admissible Galois Structures

In a Barr-exact category we have:

Definition (Janelidze[3])

Given an admissible Galois structure $(S, C, \mathcal{A}, \mathcal{P})$, an arrow f of $\mathcal{A}$ is

- a *trivial extension* if that the square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & PS(A) \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle PS(f)} \\ B & \xrightarrow[\eta_B]{} & PS(B) \end{array}$$

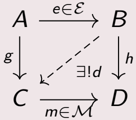
  is a pullback (where η is the unit of the adjunction $S \dashv P$);
- a *normal extension* if it is a regular epimorphism and its kernel pair projections are trivial extensions;
- a *central extension* if there exists a regular epimorphism g such that the pullback $g^*(f)$ is a trivial extension.

Factorization Systems

Definition

A *factorization system* for a category $\mathbb{C}$ is a pair of classes of arrows $(\mathcal{E}, \mathcal{M})$ such that:

- for every commutative square in $\mathbb{C}$ of the form



  there exists a unique arrow $d\colon B \to C$ such that $de = g$ and $md = h$;
- every arrow f in $\mathbb{C}$ factors as $f = me$, where $m \in \mathcal{M}$ and $e \in \mathcal{E}$.

A factorization system $(\mathcal{E}, \mathcal{M})$ is *stable* if $\mathcal{E}$ is pullback stable.

Pretorsion Theories, Galois Structures and Factorization Systems

Conditions on the Pretorsion Theories

The following part is strongly inspired by the results obtained by Everaert and Gran[4].

Definition

A functor F between protomodular categories that have both a terminal and an initial object is *protoadditive* if it preserves the terminal object, the initial object, and pullbacks along split epimorphisms.

A pretorsion theory $(\mathcal{T}, \mathcal{F})$ satisfies condition (P) if the reflector F is protoadditive (in our sense).

A pretorsion theory $(\mathcal{T}, \mathcal{F})$ satisfies condition (N) if for every diagram

$$T(K[f]) \xrightarrow{\varepsilon_{K[f]}} K[f] \xrightarrow{k} A \xrightarrow{f} B,$$

where $k = \mathrm{preker}(f)$, $k\varepsilon_{K[f]}$ is the prekernel of some arrow.

[4] *Protoadditive functors, derived torsion theories and homology*, Journal of Pure and Applied Algebra.

Conditions on the Torsion Objects T s.t. $F(T) = \mathbf{1}$

A pretorsion theory satisfies condition (C1) if, for every $A, T \in \mathbb{C}$,

$$F(T) = \mathbf{1} \text{ implies } F(A \times T) \cong F(A).$$

A pretorsion theory satisfies condition (C2) if, whenever $F(T) = \mathbf{1}$ and $A \neq \mathbf{1}$, the sequence

$$\mathbf{2} \times T \xrightarrow{\iota_A \times id_T} A \times T \xrightarrow{\pi_A} A$$

is pre-exact and $\mathbf{2} \times T \in \mathcal{T}$.

Pretorsion Theories and Galois Structures

Pretorsion theories $(\mathcal{T}, \mathcal{F})$ satisfying (C1)

$$\updownarrow \ 1{:}1$$

Precok-refl subcat $\mathcal{F}$ s.t. $F(\mathbf{1}) = \mathbf{1}$ and $i_{\mathcal{F}} \dashv F$ is admissible w.r.t. all arrows.

Idea:

$$\mathcal{T} := \{T \in \mathbb{C} \mid T = K[\eta_X] \text{ for } X \in \mathbb{C}\}.$$

Normal and Central Extensions

Moreover, if $(\mathcal{T}, \mathcal{F})$ satisfies condition (P) we have a useful characterization of central extensions.

Theorem

Consider a pretorsion theory $(\mathcal{T}, \mathcal{F})$ satisfying conditions (P) and (C1). Suppose f is a regular epimorphism, and let $\Gamma_{\mathcal{F}}$ be the Galois structure associated with the reflector F. Then, the following conditions are equivalent:

- f is a normal extension for $\Gamma_{\mathcal{F}}$;
- f is a central extension for $\Gamma_{\mathcal{F}}$;
- $K[f] \in \mathcal{F}$.

Pretorsion Theories and Factorization System

Pretorsion theories $(\mathcal{T}, \mathcal{F})$ satisfying (N) and (C2)

$$\Big\updownarrow \; 1{:}1$$

Stable factorization systems $(\mathcal{E}, \mathcal{M})$ s.t. $\mathcal{E}$ are precokernels and $\mathbf{2} \to \mathbf{1} \in \mathcal{M}$.

Idea:

$$\begin{aligned} \mathcal{E} &\coloneqq \{e \text{ precokernel } \mid K[e] \in \mathcal{T}\} \text{ and } \mathcal{M} \coloneqq \{m \mid K[m] \in \mathcal{F}\}; \\ \mathcal{T} &\coloneqq \{T \in \mathbb{C} \mid \exists t\colon T \to \mathbf{2}, t \in \mathcal{E}\} \cup \{T \in \mathbb{C} \mid \tau_T \in \mathcal{E}\} \text{ and } \mathcal{F} \coloneqq \{F \in \mathbb{C} \mid \tau_F \in \mathcal{M}\}. \end{aligned}$$

Examples

Semisimple MV-Algebras

An MV-algebra $(A, \oplus, 0, \neg)$ is a commutative monoid, such that $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.
The *radical* of A, denoted with $\operatorname{Rad}(A)$, is defined as the intersection of all maximal ideals of A.
An MV-algebra A is *semisimple* if its radical is trivial (i.e. $\operatorname{Rad}(A) = \{0\}$).

Proposition

The full subcategory of semisimple MV-algebras $s\mathbb{MV}$ is reflective in $\mathbb{MV}$. The reflector is given by

$$S(A) := A/\operatorname{Rad}(A).$$

The reflection above defines the pretorsion theory $(p\mathbb{MV}, s\mathbb{MV})$, where $p\mathbb{MV}$ denotes the full subcategory of $\mathbb{MV}$ of perfect MV-algebras; an MV-algebra A is said to be *perfect* if $A = \operatorname{Rad}(A) \cup \neg\operatorname{Rad}(A)$.

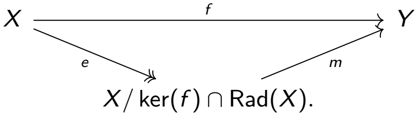
Semisimple MV-Algebras

Proposition

$(p\mathbb{MV}, s\mathbb{MV})$ satisfies conditions (P), (N), (C1), and (C2).

The associated factorization system is given by

$$\mathcal{E} := \{e\colon A \to B \in \mathrm{Arr}(\mathbb{MV}) \mid e \text{ is surjective and } \ker(e) \subseteq \mathrm{Rad}(A)\} \text{ and}$$
$$\mathcal{M} := \{m\colon A \to B \in \mathrm{Arr}(\mathbb{MV}) \mid \ker(m) \cap \mathrm{Rad}(A) = \{0\}\}.$$



A regular epimorphism s is central for the structure $\Gamma_{s\mathbb{MV}}$ if and only if $K[s] = \ker(s) \cup \neg\ker(s)$ is a semisimple MV-algebra.

Double Negation in Heyting Algebras

Given a Heyting algebra H, let $H_{\neg\neg}$ denote the set of regular elements of H. An element $x \in H$ is said to be *regular* if $\neg\neg x = x$. It is a known fact that $(H_{\neg\neg}, \vee_{\neg\neg}, \wedge, 0, 1, \Rightarrow)$ is a Boolean algebra, where $x \vee_{\neg\neg} y := \neg(\neg x \wedge \neg y)$.

Proposition

The full subcategory $\mathbb{B}\text{oole}$ is reflective in $\mathbb{H}\text{eyt}$. The reflector is given by

$$F(H) := H_{\neg\neg}.$$

The reflection above defines the pretorsion theory $(\mathbb{PD}, \mathbb{B}\text{oole})$, where $\mathbb{PD}$ denotes the full subcategory of $\mathbb{H}\text{eyt}$ of pseudo-deterministic Heyting algebras; a Heyting algebras H is said to be *pseudo-deterministic* if $\neg x = 1$ or $\neg x = 0$ for every $x \in H$.

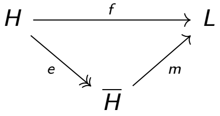
Double Negation in Heyting Algebras

Proposition

$(\mathbb{PD}, \mathbb{B}\text{oole})$ satisfies conditions (P), (N), (C1), and (C2).

The associated factorization system is given by

$$\mathcal{E} := \{e \in \mathrm{Arr}(\mathbb{H}\mathrm{eyt}) \mid e \text{ is a precokernel and } K[e] \in \mathbb{PD}\} \text{ and}$$
$$\mathcal{M} := \{m \in \mathrm{Arr}(\mathbb{H}\mathrm{eyt}) \mid K[m] \in \mathbb{B}\mathrm{oole}\}$$



where $\overline{H} := H/(\mathrm{Eq}(f) \cap \mathrm{Eq}(\eta_H))$.

A regular epimorphism s is central for the structure $\Gamma_{\mathbb{B}\mathrm{oole}}$ if and only if $K[s]$ is a Boolean algebra.

M$\mathbb{S}$et and Fix Points

For every monoid M, the category M$\mathbb{S}$et of set with a fixed action of M is a two-valued elementary topos. Given an object X of M$\mathbb{S}$et, we define the set of fix points $\mathrm{Fix}(X) := \{x \in X \mid mx = x \text{ for every } m \in M\}$.

We define two full subcategories of M$\mathbb{S}$et whose objects are

$$\mathcal{F} := \{X \in \mathrm{M\mathbb{S}et} \mid \mathrm{Fix}(X) = X\} \text{ and } \mathcal{T} := \{X \in \mathrm{M\mathbb{S}et} \mid |\mathrm{Fix}(X)| \leq 1\}.$$

Proposition

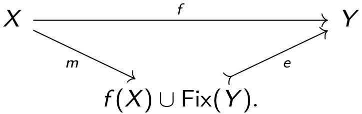
The pretorsion theory $(\mathcal{T}, \mathcal{F})$ in M$\mathbb{S}$et^{op} satisfies conditions (P), (N), (C1), and (C2).

M$\mathbb{S}$et and Fix Points

We describe the associated factorization system and the central extensions in terms of arrows of M$\mathbb{S}$et.

The associated factorization system is given by

$$\mathcal{E} := \{e\colon A \to B \in \mathrm{Arr}(\mathrm{M}\mathbb{S}\mathrm{et}) \mid e \text{ is a prekernel and } B/e(A) \in \mathcal{T}\} \text{ and}$$
$$\mathcal{M} := \{m\colon A \to B \in \mathrm{Arr}(\mathrm{M}\mathbb{S}\mathrm{et}) \mid B/m(A) \in \mathcal{F}\}$$



A monomorphism $f\colon X \to Y$ is a central extension for the structure $\Gamma_{\mathcal{F}}$ if and only if $Y/f(X) \in \mathcal{F}$ (or, equivalently, if $Y = \mathrm{Fix}(Y) \cup f(X)$).

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Thank You for Your Attention!