Decidable objects and molecular toposes

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May 2023

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F. W. Lawvere. Cohesive Toposes and Cantor's 'lauter Einsen'. Philos. Math., III. Ser. 2, No. 1, 5-15 (1994).

The extraction of zero-dimensional/discrete spaces

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‘Spaces’

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and specific connections such as:

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and specific connections such as:

1. (Connected components)
   A left adjoint ‘π_0’ with stable units (i.e. preserving finite products and ’well-behaved on fibers’).
2. (Points) A colimit preserving right adjoint.

Totally disconnected spaces (extracted from **Top**)

Let **TD** $\rightarrow$ **Top** be the full subcategory of totally disconnected topological spaces (only connected subsets are single points).

It is reflective. The left adjoint π_0 sends a space X to the t.d. space $\pi_0 X$ of connected components.

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Theorem (Stable units)

π_0 : **Top** $\rightarrow$ **TD** *preserves pullbacks over t.d. spaces.*
In particular, π_0 *preserves finite products.*

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(Note: π_0 : **Top** $\to$ **TD** preserves all small products.)

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Intuition: $\pi_0 X$ is the totally separated space of 'quasi-components'.

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(Janelidze 2009): $\pi_0 :$ **Top** $\to$ **TS** does not have stable units (is not even semi-left-exact).

Definition

A geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is hyperconnected if $p^* : \mathcal{S} \to \mathcal{E}$ is fully faithful and the counit β of $p^* \dashv p_*$ is monic.

Intuition:

Hyperconnected geometric morphisms

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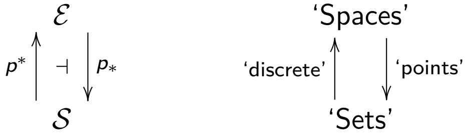
Intuition: $\mathcal{E}$ is a category of spaces,
$p^* : \mathcal{S} \to \mathcal{E}$ is the full subcategory of discrete spaces,
$p_* X$ is the set of points of X
$\beta_X : p^*(p_* X) \to X$ is the discrete subspace of points of X.



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$$\begin{array}{ccc} X & \longrightarrow & \mathbf{2}^{(\mathbf{2}^X)} \\ \downarrow & & \downarrow {\scriptstyle \mathbf{2}^{\beta}} \\ \pi_0 X & \longrightarrow & \mathbf{2}^{p^*(p_*(\mathbf{2}^X))} \end{array}$$

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Exponential ideal implies π_0 preserves finite products. □

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Corollary

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Corollary

*If $p : \mathcal{E} \to$ **Set** is hyperconnected essential then $p_! = \pi_0 \dashv p_*$ has stable units.*

Proof.

Because every essential g.m. over **Set** is molecular. □

Notice!

$$\mathbf{TS} \longrightarrow \mathbf{Top} \qquad\qquad \mathcal{S} \xrightarrow{p^*} \mathcal{E}$$

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle \sigma} & & \\ \pi_0 X & \longrightarrow & \prod_{\mathbf{Top}(X,\mathbf{2})} \mathbf{2} \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & \mathbf{2}^{(\mathbf{2}^X)} \\ \downarrow & & \downarrow{\scriptstyle \mathbf{2}^\beta} \\ \pi_0 X & \longrightarrow & \mathbf{2}^{p^*(p_*(\mathbf{2}^X))} \end{array}$$

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Essentially the same proof but:

1. $\pi_0 \dashv (\mathbf{TS} \to \mathbf{Top})$ does not have stable units.
2. $p : \mathcal{E} \to \mathbf{Set}$ hyperconnected with p^* exponential ideal, $\pi_0 \dashv p^*$ has stable units.

A very basic (new?) fact

Recall: an object X in an extensive category is decidable if the diagonal $\Delta : X \to X \times X$ is complemented.

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Proposition [M'2022]

Let $\mathcal{E}$ and $\mathcal{S}$ be extensive with finite products, and let $\Psi : \mathcal{E} \to \mathcal{S}$ be a finite-coproduct preserving. Then Ψ preserves finite products if and only if it preserves pullbacks over decidable objects.

Proof

For the non trivial direction assume that the square on the left

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & Y \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[g]{} & S \end{array} \qquad\qquad \begin{array}{ccc} P & \longrightarrow & S \\ {\scriptstyle \langle \pi_0, \pi_1 \rangle}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times Y & \xrightarrow[g \times h]{} & S \times S \end{array}$$

is a p.b. in $\mathcal{E}$ with decidable S. I.e. the right square above is a p.b.

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$$\begin{array}{ccc} \Psi P & \longrightarrow & \Psi S \\ {\scriptstyle \langle \Psi\pi_0, \Psi\pi_1 \rangle}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \Psi X \times \Psi Y & \xrightarrow[\Psi g \times \Psi h]{} & \Psi S \times \Psi S \end{array} \qquad\qquad \begin{array}{ccc} \Psi P & \xrightarrow{\Psi\pi_1} & \Psi Y \\ {\scriptstyle \Psi\pi_0}\downarrow & & \downarrow{\scriptstyle \Psi h} \\ \Psi X & \xrightarrow[\Psi g]{} & \Psi S \end{array}$$

is a p.b. in $\mathcal{S}$. That is, the square on the right above is a p.b..

Stable units almost for free

Let $\mathcal{E}$ be extensive with finite products.

An object X in $\mathcal{E}$ is decidable if the diagonal $\Delta : X \to X \times X$ is complemented.

Corollary

If $\mathrm{Dec}\mathcal{E} \to \mathcal{E}$ has a finite-product preserving left adjoint then the reflection has stable units.

Stable units from product-preservation

Theorem (M'2022)

If $\mathcal{S}$ is a Boolean topos then, for every connected essential geometric morphism $p : \mathcal{E} \to \mathcal{S}$ such that the leftmost adjoint $p_!$ preserves finite products, p is molecular and

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$$\begin{array}{c} \mathcal{E} \\ p_! \downarrow \dashv \uparrow p^* \dashv \downarrow p_* \\ \mathrm{Dec}\mathcal{E} \end{array}$$

$\pi_0 = p_!$ preserving finite products.

Pre-cohesive geometric morphisms

Definition

A hyperconnected $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive if $p^* \dashv p_*$ extends to a string of adjoints

$$p_! \dashv p^* \dashv p_* \dashv p^!$$

such that $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products.

Intuition: Components $\dashv$ Discrete $\dashv$ Points $\dashv$ Codiscrete.

Examples

Proposition [Johnstone 2011]

A bounded geometric morphism $p : \mathcal{E} \to \mathbf{Set}$ is pre-cohesive iff $\mathcal{E}$ has a connected and locally connected site of definition $(\mathcal{C}, J)$ such that every object of $\mathcal{C}$ has a point.

"The contrast of cohesion $\mathcal{E}$ with non-cohesion $\mathcal{S}$ can be expressed by geometric morphisms $p : \mathcal{E} \to \mathcal{S}$ but that contrast can be made relative, so that $\mathcal{S}$ itself may be an 'arbitrary' topos.[...]

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For example, in a case $\mathcal{E}$ of algebraic geometry wherein spaces of all dimensions exist, $\mathcal{S}$ is usefully taken as a corresponding category of zero-dimensional spaces such as the Galois topos (of Barr-atomic sheaves on finite extensions of the ground field)". [L'07]

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(Streicher) $\mathcal{S}^{({\Delta_1}^{op})} \to \mathcal{S}$ is pre-cohesive.

Pre-cohesive maps over Boolean toposes are molecular

Corollary

If $\mathcal{S}$ is Boolean and $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive then p is molecular.
(So it has stable units.)
In this case,

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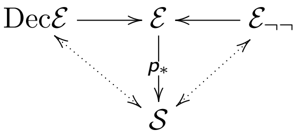

Are pre-cohesive maps molecular?

We don't know.

Negative evidence

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[Garner-Streicher TAC 2021] built an essential, local map whose inverse image is an exponential ideal that is not l.c.
(Not pre-cohesive because it is not hyperconnected.)

Positive evidence

Corollary

If $\mathcal{S}$ is Boolean and $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive then p is molecular. (So it has stable units.)

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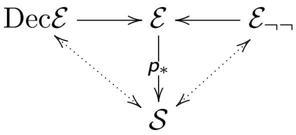

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What are the toposes $\mathcal{S}$ such that every essential g.m. with codomain $\mathcal{S}$ is molecular?

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Theorem (Hemelaer 2022)

If X is T_1 then every essential g.m. with codomain $\mathrm{Sh}X$ is molecular.

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Bibliography I

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