A Doctrinal View of Logic

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May 24th, 2023

Doctrines

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Definition

Let $\mathbb{C}$ be a category with finite products and let **Pos** be the category of partially-ordered sets and monotone functions. A *doctrine* is a functor $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$.

$$\mathbb{C}^{\mathrm{op}} \xrightarrow{\;P\;} \mathbf{Pos}$$

$$\begin{array}{ccc} B & & P(B) \\ \uparrow f & & \downarrow P(f) \\ A & & P(A) \end{array}$$

Examples

(a) The functor $\mathscr{P} : \mathrm{Set}^{\mathrm{op}} \to \mathbf{Pos}$, sending each set in the poset of its subsets, ordered by inclusion, and each function $f : A \to B$ to the inverse image $f^{-1} : \mathscr{P}(B) \to \mathscr{P}(A)$ is a doctrine.

(b) For a given category $\mathbb{C}$ with finite limits, the functor $\mathrm{Sub}_{\mathbb{C}} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ sending each object to the poset of its subobjects in $\mathbb{C}$ and each arrow $f : X \to Y$ to the pullback function $f^* : \mathrm{Sub}_{\mathbb{C}}(Y) \to \mathrm{Sub}_{\mathbb{C}}(X)$, is a doctrine.

Doctrines

Examples

(c) For a given theory $\mathcal{T}$ on a one-sorted first-order language $\mathcal{L}$, define the category $\mathbb{C}\mathrm{tx}_{\mathcal{L}}$ of contexts: an object is a finite list of distinct variables and an arrow between two lists $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_m)$ is given by an m-tuple of terms in the context $\vec{x}$:

$$(t_1(\vec{x}), \ldots, t_m(\vec{x})) : (x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$$

The functor $\mathrm{LT}^{\mathcal{L}}_{\mathcal{T}} : \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \to \mathbf{Pos}$ sends each list $\vec{y}$ of variables to the poset reflection of well-formed formulae ordered by provable consequence in $\mathcal{T}$.

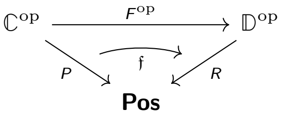
$$\mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \xrightarrow{\mathrm{LT}^{\mathcal{L}}_{\mathcal{T}}} \mathbf{Pos}$$

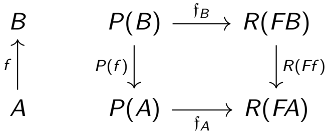
$$\begin{array}{ccccc} \vec{y} & & \mathrm{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{y}) & \ni \alpha(\vec{y}) \vdash_{\mathcal{T}} \beta(\vec{y}) \\ {\scriptstyle \vec{t}(\vec{x})}\uparrow & & \downarrow{\scriptstyle [\vec{t}(\vec{x})/\vec{y}]} & \downarrow \\ \vec{x} & & \mathrm{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{x}) & \ni \alpha(\vec{t}(\vec{x})) \end{array}$$

Doctrines homomorphisms

Definition

A *doctrine homomorphism* between $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ and $R : \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ is a pair $(F, \mathfrak{f})$ where $F : \mathbb{C} \to \mathbb{D}$ is a functor that preserves finite products and $\mathfrak{f} : P \xrightarrow{\cdot} R \circ F^{\mathrm{op}}$ is a natural transformation.





Additional structures

Definition

(a) A *primary doctrine* $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is a doctrine such that for each object A in $\mathbb{C}$, the poset $P(A)$ has finite meets, and the related operations $\wedge : P \times P \dot{\to} P$ and $\top : \mathbf{1} \dot{\to} P$ yeld natural transformations.

(b) is *implicational* if for any object A, the poset $P(A)$ is cartesian closed, and the related operations $\wedge : P \times P \dot{\to} P$, $\top : \mathbf{1} \dot{\to} P$, $\to : P^{\mathrm{op}} \times P \dot{\to} P$ yeld natural transformations;

(c) *is bounded* if for any object A, the poset $P(A)$ has a top and a bottom element, and the related operation, $\top : \mathbf{1} \dot{\to} P$ and $\bot : \mathbf{1} \dot{\to} P$ yeld natural transformations;

(d) is *Boolean* if for any object A, the poset $P(A)$ is a Boolean algebra, and the related operations $\wedge : P \times P \dot{\to} P$, $\top : \mathbf{1} \dot{\to} P$, $\to : P^{\mathrm{op}} \times P \dot{\to} P$, $\vee : P \times P \dot{\to} P$, $\bot : \mathbf{1} \dot{\to} P$ yeld natural transformations;

Additional structures

Definition

(e) A primary doctrine $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is *existential* if for any pair of objects B, C of $\mathbb{C}$, the map $P(\mathrm{pr}_1) : P(C) \to P(C \times B)$ has a left adjoint

$$\exists_C^B : P(C \times B) \to P(C),$$

which is natural in C; moreover, the adjunction $\exists_C^B \dashv P(\mathrm{pr}_1)$ satisfies the Frobenius reciprocity, i.e. for any $\alpha \in P(C \times B)$ and $\beta \in P(C)$ the equality

$$\exists_C^B(\alpha \wedge P(\mathrm{pr}_1)(\beta)) = \exists_C^B(\alpha) \wedge \beta$$

holds.

Definition

A doctrine homomorphism $(F, \mathfrak{f})$ is primary (resp. implicational, bounded, Boolean, existential) if $\mathfrak{f}$ preserves the structure.

Additional structures: examples

Examples

(a) The doctrine $\mathscr{P} : \mathrm{Set}^{\mathrm{op}} \to \mathbf{Pos}$ is Boolean and existential.

$$\mathscr{P}(C \times B) \underset{\mathrm{pr}_1^{-1}}{\overset{\mathrm{pr}_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathscr{P}(C)$$

(b) For a given category $\mathbb{C}$ with finite limits, the doctrine $\mathrm{Sub}_{\mathbb{C}} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ is primary. If $\mathbb{C}$ is regular, $\mathrm{Sub}_{\mathbb{C}}$ is existential.

(c) The doctrine $\mathrm{LT}^{\mathcal{L}}_{\mathcal{T}} : \mathbb{C}\mathrm{tx}^{\mathrm{op}}_{\mathcal{L}} \to \mathbf{Pos}$ is Boolean and existential.

$$\mathrm{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{x}, \vec{y}) \overset{\exists y_1 \ldots \exists y_m}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}}} \mathrm{LT}^{\mathcal{L}}_{\mathcal{T}}(\vec{x})$$

Henkin's proof

Main goal

Henkin's Theorem: L. Henkin, 1949[1]

Let $\mathcal{T}$ be a theory in a first-order language $\mathcal{L}$. If $\mathcal{T}$ is consistent, then $\mathcal{T}$ has a model.

Consistent theory $\mathcal{T}$ in a first-order language $\mathcal{L}$	Suitable doctrine $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$
Model of $\mathcal{T}$	Suitable doctrine homomorphism from P to $\mathscr{P}_* : \mathrm{Set}_*^{\mathrm{op}} \to \mathbf{Pos}$

[1] Leon Henkin. *The completeness of the first-order functional calculus*, The Journal of Symbolic Logic.

Idea of the proof

Steps of Henkin's proof, adapted to doctrines:

Consistent first-order theory $\mathcal{T}$ in the language $\mathcal{L}$.

1. Extend the language with a suitable amount of constants;
2. extend the theory with formulae of the kind $\exists x \varphi(x) \to \varphi(c)$;
3. show that consistency still holds;
4. define a model whose underlying set is given by the closed terms of the extended language.

P bounded existential implicational doctrine, with non-trivial fibers and with a small base category.

1.a Add a constant to the language;

1.b add a suitable amount of constants to the language;

2.a extend the theory with an axiom;

2.b extend the theory with formulae of the kind $\exists x \varphi(x) \to \varphi(c)$;

3. show that consistency still holds;
4. define a model whose base functor is given by $\mathrm{Hom}(\mathbf{t}, -)$.

Adding a constant and an axiom to a doctrine

Adding a constant

Let $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a doctrine and X be a fixed object in $\mathbb{C}$.

$$\mathbb{C}_X^{\mathrm{op}} \xrightarrow{\;P_X\;} \mathbf{Pos}$$

$$\begin{array}{ccccc} B & & B & & P(X \times B) \\ \uparrow{\scriptstyle f} & \longleftrightarrow & \rightsquigarrow{\scriptstyle f} & & \downarrow{\scriptstyle P(\langle \mathrm{pr}_1, f\rangle)} \\ X \times A & & A & & P(X \times A) \end{array}$$

In particular, there exists $\mathbf{t} \rightsquigarrow X$ a arrow in $\mathbb{C}_X$, corresponding to $\mathrm{id}_X : X \to X$.

Adding a constant

$$\begin{array}{ccc} \mathbb{C}^{\mathrm{op}} & \xrightarrow{\quad F_X^{\mathrm{op}} \quad} & \mathbb{C}_X^{\mathrm{op}} \\ & {\scriptstyle P} \searrow \; \overset{\mathfrak{f}^{(X)}}{\dashrightarrow} \; \swarrow {\scriptstyle P_X} & \\ & \mathbf{Pos} & \end{array}$$

$$\mathbb{C} \xrightarrow{F_X} \mathbb{C}_X$$

$$\begin{array}{ccccc} A & & A & & X \times A \\ {\scriptstyle f}\downarrow & & \rightsquigarrow & \longleftrightarrow & \downarrow{\scriptstyle f\mathrm{pr}_2} \\ B & & B & & B \end{array}$$

$$P(A) \xrightarrow{\mathfrak{f}_A^{(X)}} P_X(A) = P(X \times A)$$

$$\alpha \longmapsto P(\mathrm{pr}_2)(\alpha)$$

Remarks

- If the doctrine P is primary (resp. implicational, bounded, Boolean, existential), then also P_X and $(F, \mathfrak{f}^{(X)})$ are primary (implicational, bounded, Boolean, existential);
- $(F, \mathfrak{f}^{(X)})$ has a universal property.

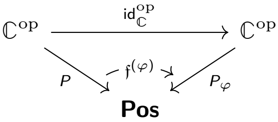
Adding an axiom

Let $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a primary doctrine and φ be a fixed element in $P(\mathbf{t})$.

$$\mathbb{C}^{\mathrm{op}} \xrightarrow{\quad P_\varphi \quad} \mathbf{Pos}$$

$$\begin{array}{ccc} B & \qquad & P(B)_{\downarrow P(!_B)\varphi} \\ \uparrow f & & \downarrow P(f) \\ A & & P(A)_{\downarrow P(!_A)\varphi} \end{array}$$

Adding an axiom



$$P(A) \xrightarrow{\mathfrak{f}_A^{(\varphi)}} P_\varphi(A) = P(A)_{\downarrow P(!_A)\varphi}$$

$$\alpha \longmapsto \alpha \wedge P(!_A)\varphi$$

In particular, $\mathfrak{f}_{\mathbf{t}}^{(\varphi)} : \varphi \mapsto \varphi$, so that φ is sent to the top element of $P_\varphi(\mathbf{t})$.

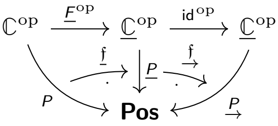
Remarks

- The doctrine P_φ and the homomorphism $(F, \mathfrak{f}^{(\varphi)})$ are primary;
- if the doctrine P is implicational (resp. bounded, Boolean, existential), then also P_X and $(F, \mathfrak{f}^{(\varphi)})$ are implicational (bounded, Boolean, existential);
- $(F, \mathfrak{f}^{(\varphi)})$ has a universal property.

Rich doctrines and Henkin's Theorem

Richness

Let $P : \mathbb{C}^{\mathrm{op}} \to \textbf{Pos}$ be an implicational existential doctrine, with a small base category.



The doctrine $\underline{P}$ has a suitable amount of added constants. In the doctrine $\underrightarrow{P}$ suitable formulas of the kind $\exists x \varphi(x) \to \varphi(c)$ are made true. All additional structures are preserved.

Definition

Let $R : \mathbb{D}^{\mathrm{op}} \to \textbf{Pos}$ be an existential doctrine. Then R is *rich* if for all $A \in \mathrm{ob}\mathbb{D}$ and for all $\sigma \in R(A)$ there exists a $\mathbb{D}$-arrow $d : \mathbf{t} \to A$ such that $\exists_{\mathbf{t}}^{A} \sigma \leq R(d)\sigma$.

Richness

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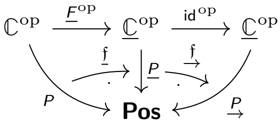

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Definition

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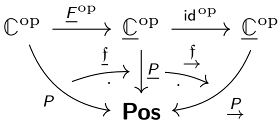
Example

The subsets doctrine $\mathscr{P} : \mathrm{Set}^{\mathrm{op}} \to \mathbf{Pos}$ is not rich, since there exists no arrow $\mathbf{t} \to \emptyset$. However, we can remove the empty set from the base category and consider the doctrine $\mathscr{P}_* : \mathrm{Set}_*^{\mathrm{op}} \to \mathbf{Pos}$, which is rich.

Richness

Let $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be an implicational existential doctrine, with a small base category.



The doctrine $\underline{P}$ has a suitable amount of added constants. In the doctrine $\underset{\to}{P}$ suitable formulas of the kind $\exists x \varphi(x) \to \varphi(c)$ are made true. All additional structures are preserved.

Definition

Let $R : \mathbb{D}^{\mathrm{op}} \to \mathbf{Pos}$ be an existential doctrine. Then R is *rich* if for all $A \in \mathrm{ob}\mathbb{D}$ and for all $\sigma \in R(A)$ there exists a $\mathbb{D}$-arrow $d : \mathbf{t} \to A$ such that $\exists^A_{\mathbf{t}} \sigma \leq R(d)\sigma$.

- The doctrine homomorphism $(F, \mathfrak{f}) : P \to \underset{\to}{P}$ is implicational existential;
- $\underset{\to}{P}$ is rich;
- $(F, \mathfrak{f})$ has a weak universal property.

Consistency

Definition

A bounded doctrine P is *consistent* if $\top_{\mathbf{t}} \nleq \bot_{\mathbf{t}}$ in $P(\mathbf{t})$.

Proposition

Let $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a bounded existential implicational doctrine such that each fiber is non-trivial, and the base category $\mathbb{C}$ is small, then the doctrine $\underrightarrow{P}$ is consistent.

Idea of the proof:

- Prove the Proposition above with *Boolean* instead of *implicational*;
- Use the weak universal property and the Boolean completion to prove the Proposition in the implicational setting.

$$\begin{array}{ccc} P & \longrightarrow & P_{\neg\neg} \\ \downarrow & & \downarrow \\ \underrightarrow{P} & \longrightarrow & \underrightarrow{P_{\neg\neg}} \end{array}$$

Filters

Definition

Let A be an inf-semilattice. A subset $\nabla \subseteq A$ is a *filter* if the following properties hold:

- $\top \in \nabla$;
- if $a \in \nabla$ and $a \leq b$, then $b \in \nabla$;
- if $a, b \in \nabla$, then $a \wedge b \in \nabla$.

A filter ∇ is *proper* if $\nabla \neq A$.

A filter ∇ is a *maximal filter* if it is maximal with respect to the inclusion, meaning that $\nabla \neq A$ and, whenever $\nabla \subsetneq \nabla'$ where ∇' is a filter, then $\nabla' = A$.

Lemma

Given a proper filter ∇ of a bounded implicative inf-semilattice A, there exists a maximal filter $U \supseteq \nabla$.

A model of a rich doctrine

Let $P : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be a bounded consistent existential implicational **rich** doctrine. Let $\nabla \subseteq P(\mathbf{t})$ be a **maximal filter** and $P/\nabla : \mathbb{C}^{\mathrm{op}} \to \mathbf{Pos}$ be the relative quotient doctrine.

We build a model of P/∇ in the doctrine $\mathscr{P}_* : \mathrm{Set}_*^{\mathrm{op}} \to \mathbf{Pos}$, i.e. a doctrine homomorphism $(\Gamma, \mathfrak{g}) : P/\nabla \to \mathscr{P}_*$, preserving the bounded existential implicational structure.

Define $\Gamma := \mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, -) : \mathbb{C} \to \mathrm{Set}_*$. Then, for a given $X \in \mathrm{ob}\mathbb{C}$, let $\mathfrak{g}_X : P/\nabla(X) \to \mathscr{P}_*(\mathrm{Hom}_{\mathbb{C}}(\mathbf{t}, X))$ be:

$$\mathfrak{g}_X[\varphi] = \{c : \mathbf{t} \to X \mid P(c)\varphi \in \nabla\}.$$

Proposition

The pair $(\Gamma, \mathfrak{g}) : P/\nabla \to \mathscr{P}_*$ is a bounded **existential implicational** morphism.

Henkin's theorem

Theorem

Let P be a bounded existential implicational doctrine, with non-trivial fibers and with a small base category. Then there exists a bounded existential implicational model of P in the doctrine of subsets $\mathscr{P}_* : \mathrm{Set}_*^{\mathrm{op}} \to \mathbf{Pos}$.

Proof. Thanks to the construction seen before, we get a morphism $(F, \mathfrak{f}) : P \to \underset{\to}{P}$ that preserves bounded implicational existential structure; moreover the doctrine $\underset{\to}{P}$ is consistent and rich. So $\underset{\to}{P}$ is a bounded, existential, implicational doctrine, rich and consistent, so that we can chose a maximal filter $\nabla \subseteq \underset{\to}{P}(\mathbf{t})$ and take the quotient over it. Hence define as before the model $(\Gamma, \mathfrak{g})$ of such quotient. The composition

$$P \xrightarrow{(F,\mathfrak{f})} \underset{\to}{P} \xrightarrow{(\mathrm{id},\mathfrak{q})} \underset{\to}{P}/\nabla \xrightarrow{(\Gamma,\mathfrak{g})} \mathscr{P}_*$$

is a model of P, preserving all said structure. □

Thank you for your attention!