What should Strong Vector Spaces be?

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Based strong vector spaces

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  $\Gamma \hookrightarrow \mathbf{k}(\Gamma, \mathcal{F})$, identify γ with $\delta_\gamma(\gamma') = \begin{cases} 1 & \text{if } \gamma = \gamma' \\ 0 & \text{otherwise} \end{cases}$

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- $(x_i)_{i \in I} \in \mathbf{k}(\Gamma, \mathcal{F})$ is *summable* iff

$$\forall \gamma \in \Gamma |\{i \in I : \gamma \in \operatorname{supp} x_i\}| < \aleph_0$$
$$\bigcup \{\operatorname{supp} x_i : i \in I\} \in \mathcal{F}.$$

in which case $\left(\sum_{i \in I} x_i\right)(\gamma) = \sum_{i \in I} x_i(\gamma)$.

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  in which case $\left(\sum_{i\in I} x_i\right)(\gamma) = \sum_{i\in I} x_i(\gamma)$.
- $f : \mathbf{k}(\Gamma, \mathcal{F}) \to \mathbf{k}(\Delta, \mathcal{G})$ linear, is *strongly linear* iff

$$\forall (x_i)_{i\in I} \in \mathbf{k}(\Gamma, \mathcal{F})^I \text{ summable } \to \begin{cases} (f x_i)_{i\in I} \in \mathbf{k}(\Delta, \mathcal{G})^I \text{ summable} \\ \sum_{i\in I} f x_i = f \sum_{i\in I} x_i \end{cases}$$

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$B\Sigma\mathrm{Vect}$ is the category whose objects are the $\mathbf{k}(\Gamma, \mathcal{F})$s and whose arrows are strong linear maps between them.

Examples - why do we care

If $<$ is a total order on Γ and $\mathcal{F} \subseteq WO(\Gamma, <) = \{S \subseteq \Gamma : S \text{ is well ordered}\}$ then $\mathbf{k}(\Gamma, \mathcal{F})$ has a valuation

$$v : \mathbf{k}(\Gamma, \mathcal{F})^{\neq 0} \to \Gamma \qquad v(x) = \min \operatorname{supp} x.$$

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Fact: $\mathbf{k}(\Gamma, WO(\Gamma))$ is isomorphic to the $\mathcal{H}$-injective hull of any $(V, v : V^{\neq 0} \to \Gamma)$ with 1-dimensional ribs, where $\mathcal{H}$ is the class of immediate extensions of valued vector spaces.

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If $(\Gamma, <, \cdot) \in oAb$, then on $\mathbf{k}(\Gamma, WO(\Gamma))$ there is a unique product extending the product of Γ along $\gamma \mapsto \tilde{\gamma} := \delta_{\gamma,-}$ which is strongly linear in both arguments.

Example 1

- if $\Gamma = x^{\mathbb{Z}}$, then $\mathbf{k}(\Gamma, WO(\Gamma)) = \mathbf{k}((x))$.
- if $\Gamma = x^{\mathbb{Q}}$ and $\mathcal{F}$ consists of the well ordered subsets generating a fin.gen. subgroup then $\mathbf{k}(\Gamma, \mathcal{F})$ is the field of Puiseux series.

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Proposition 1

The assignment below defines a $\mathbf{k}$*-additive functor* $B\mathrm{Vect}^{\mathrm{op}} \to B\Sigma\mathrm{Vect}$ *which is fully faithful.*

$$\begin{array}{ccc} \mathbf{k}^{\oplus I} & & \mathbf{k}(I, \mathcal{P}(I)) = \mathbf{k}^I \cong (\mathbf{k}^{\oplus I})^* \\ \big\downarrow f & \mapsto & f^* \big\uparrow \\ \mathbf{k}^{\oplus J} & & \mathbf{k}(J, \mathcal{P}(J)) = \mathbf{k}^J \cong (\mathbf{k}^{\oplus J})^* \end{array}$$

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Remark: $(x_i)_{i \in I} \in \mathbf{k}(\Gamma, \mathcal{F})^I$ is summable iff $x_i = f\delta_i$ for a (unique) s.l. $f : \mathbf{k}^I \to \mathbf{k}(\Gamma, \mathcal{F})$.

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Remark: $(x_i)_{i \in I} \in \mathbf{k}(\Gamma, \mathcal{F})^I$ is summable iff $x_i = f\delta_i$ for a (unique) s.l. $f : \mathbf{k}^I \to \mathbf{k}(\Gamma, \mathcal{F})$. Notice $\delta_i \in \mathbf{k}^I$ corresponds to the i-th-coefficient-selecting functional $\delta_i \in \mathrm{Vect}(\mathbf{k}^{\oplus I}, \mathbf{k})$.

Reasonble Cats of strong vector spaces (i)

$\iota : \mathrm{Vect}^{\mathrm{op}} \to \mathcal{C}$ fully faithful $\mathbf{k}$-additive functor. For it to be a category of strong vector spaces we would like that:

- $\mathcal{C}(\iota\mathbf{k}, -)$ is faithful
- $\{$summable I-families in $\mathcal{C}(\iota\mathbf{k}, X)\} \sim \mathcal{C}(\iota(\mathbf{k}^{\oplus I}), X)$
  - $(x_i)_{i\in I} \in \mathcal{C}(\iota\mathbf{k}, X)^I$ should be summable iff $x_i = f \cdot \iota(\delta_i)$ for some $f \in \mathcal{C}(\iota(\mathbf{k}^{\oplus I}), X)$ in which case $\sum_{i\in I} x_i \cdot k_i = \mathcal{C}(\iota\mathbf{k}, f) \cdot (k_i)_{i\in I}$.

$$\mathcal{C}(\iota(\mathbf{k}^{\oplus I}), X) \xrightarrow{\mathcal{C}(\iota(\delta_i), X)_{i\in I}} \mathcal{C}(\iota\mathbf{k}, X)^I$$

  - we want the above map to be one-to-one
- each $\bar{f} : \mathcal{C}(\iota\mathbf{k}, X) \to \mathcal{C}(\iota\mathbf{k}, Y)$ has the form $\mathcal{C}(\iota\mathbf{k}, f)$ iff it preserves summability and sums.

Reasonble Cat.s of strong vector spaces (ii)

Definition

A fully faithful and locally small $\mathbf{k}$-additive extension $\iota : \mathrm{Vect}^{\mathrm{op}} \to \mathcal{C}$ is a *reasonable cat. of strong vector spaces* if

- A1 ι is dense
- A2 $\iota\mathbf{k}$ is a separator
- A3 $\mathcal{C}(\iota(\delta_i), X)_{i \in I} : \mathcal{C}(\iota(\mathbf{k}^{\oplus I}), X) \to \mathcal{C}(\iota(\mathbf{k}), X)^I$ is one-to-one for every set I.

Example: $B\Sigma\mathrm{Vect}$ is a reasonable cat. of strong vector spaces.

Remark: $(A2)$ is redundant.

Remark: $(A1)$ implies that $\mathcal{C}$ is equivalent to a full subcategory (extending the subcategory of representables) of the category $[\mathrm{Vect}, \mathrm{Vect}]_{\mathbf{k}}$ of $\mathbf{k}$-additive functors $F : \mathrm{Vect} \to \mathrm{Vect}$ and natural transformations.

Reasonble Cat.s of strong vector spaces (ii)

$$\mathcal{C} \xrightarrow{\mathcal{C}(\iota,-)} [\mathrm{Vect}, \mathrm{Vect}]_{\mathbf{k}} \xrightarrow{U_*} \mathrm{Psh}(\mathrm{Vect}^{\mathrm{op}})$$

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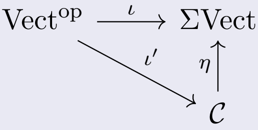
- A3 thus says $\mathrm{Nat}\left(\dfrac{\mathrm{Vect}(\mathbf{k}, -)^I}{\mathrm{Vect}(\mathbf{k}, -)^{\oplus I}}, X\right) = 0.$

The category of all strong vector spaces

Theorem 2

There is a universal r.c.s.v. $\iota : \mathrm{Vect}^{\mathrm{op}} \to \Sigma\mathrm{Vect}$*, i.e. with the property that for every r.c.s.v.* $\iota' : \mathrm{Vect}^{\mathrm{op}} \to \mathcal{C}$ *there is a (unique up to a unique isomorphism)* $\eta : \mathcal{C} \to \Sigma\mathrm{Vect}$ *s.t.* $\iota \cong \eta \circ \iota'$



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Moreover up to equivalence $\Sigma\mathrm{Vect}$ *is*

$$\left\{ X \in [\mathrm{Vect}, \mathrm{Vect}]_{\mathbf{k}} : \forall \lambda \in \mathrm{Card},\ \mathrm{Nat}\left(\frac{\mathrm{Vect}(\mathbf{k}, -)^{\lambda}}{\mathrm{Vect}(\mathbf{k}, -)^{\oplus \lambda}}, X \right) = 0 \right\}.$$

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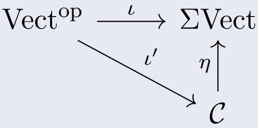

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$X \in \mathrm{Ind}(\mathrm{Vect}^{\mathrm{op}})$ iff its category of elements is finally small.

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In particular $\Sigma\mathrm{Vect}$ is reflective in $\mathrm{Ind}(\mathrm{Vect}^{\mathrm{op}})$ and hence small-bicomplete.

Arities for strong vector spaces

Definition

Say that $X \in [\mathrm{Vect}, \mathrm{Vect}]_{\mathbf{k}}$ is λ-ary if it is a left Kan extension of a $\mathbf{k}$-additive functor defined on the category Vect_λ of vector spaces of dimension $< \lambda$.

Remark: X is small if and only if it is λ-ary for some λ.

Lemma 3

$X \in \Sigma\mathrm{Vect} \subseteq [\mathrm{Vect}, \mathrm{Vect}]_{\mathbf{k}}$ is λ-ary iff whenever $(x_i)_{i \in I} \in (X\mathbf{k})^I$ is summable one has $|\{i \in I : x_i \neq 0\}| < \lambda$.

$\to$ (by a simple pigeonhole-argument) every $X \in \Sigma\mathrm{Vect}$ is $|X\mathbf{k}|^{++}$-ary.

About the reflection (i)

The following are equivalent for $X \leq Y \in \Sigma\mathrm{Vect}$

- the inclusion $X \leq Y$ reflects summability and sums of summable families (it is a "closed Σ-embedding")
- the inclusion $X \leq Y$ is right orthogonal to the natural inclusion $a_\lambda : \mathrm{Vect}(\mathbf{k}, -)^{\oplus\lambda} \leq \mathrm{Vect}(\mathbf{k}, -)^\lambda$ for every cardinal λ.
- $X \leq Y$ is a kernel in $\Sigma\mathrm{Vect}$.

For $\mathcal{M} = {}^\perp(\mathsf{Epi} \cup \{a_\lambda : \lambda \in \mathbf{Card}\})$, $(\mathcal{M}^\perp, \mathcal{M})$ is an orthogonal factorization system on $\mathrm{Ind}(\mathrm{Vect}^{\mathrm{op}})$ and the $\mathcal{M}$-factor of $f : X \to Y$ can be computed as $\mathcal{S}^\infty(f) := \ker(Y \to \mathcal{R}^\infty(\mathrm{Coker} f))$ where $\mathcal{R}^\infty$ is the postcomposition of the reflector with the inclusion $\mathcal{R}^\infty : \mathrm{Ind}(\mathrm{Vect}^{\mathrm{op}}) \to \Sigma\mathrm{Vect} \hookrightarrow \mathrm{Ind}(\mathrm{Vect}^{\mathrm{op}})$.

About the reflection (ii)

If $(\mathcal{R}, \sigma)$ is the obvious approximate reflector

$$\sigma_X : X \to \mathcal{R}(X) := \mathrm{Coker}\big(\{g : \mathrm{Coker}\, a_\lambda \to X\}_{\lambda \in \mathbf{Card}}\big)$$

the corresponding approximate $\mathcal{S}$ (restricted to $\Sigma\mathrm{Vect}$) performs the operation of sending an arrow $f : X \to Y$ to a subobject of Y whose points are all infinite sums of summable families in the image of X. This in particular motivates the question: what is the minimum ordinal α such that $\mathcal{R}^\alpha X = \mathcal{R}^\infty X$. It is easy to see that if X is λ-ary then $\mathcal{R}^\lambda = \mathcal{R}^\infty$.

Theorem 4

If X is $\aleph_1$-ary then $\mathcal{R}^\infty X = \mathcal{R}X$.

Lemma 5

If $\dim V = \aleph_0$, $H < V^*$ has the form $Span\{\delta_b^B : b \in B\}$ for a basis B of V if and only if $\dim H = \aleph_0$ and H is separative (i.e. $\bigcap_{\xi \in H} \ker \xi = 0$).

$\mathrm{Ind}(\mathrm{Vect}^{\mathrm{op}})$ is abelian and has a natural monoidal closed-structure induced by the one on $\mathrm{Vect}^{\mathrm{op}}$.

$$(X \hat{\otimes} Y)(V) \cong \int^{H_0} \int^{H_1} X(H_0) \otimes Y(H_1) \otimes \mathrm{Vect}(H_0 \otimes H_1, V)$$

$$\mathrm{Hom}(X, Y)(V) \cong \int_{H_0} \int^{H_1} \mathrm{Vect}\big(XH_0, YH_1 \otimes \mathrm{Vect}(H_1, H_0 \otimes V)\big)$$

They both restrict to $\Sigma\mathrm{Vect}$ (no reflection needed for $-\hat{\otimes}-$!).

The two operations also restrict to $B\Sigma\mathrm{Vect}$ with fairly explicit descriptions.

Topology

A linearly topologized vector space is a topological vector space (V, τ) whose topology has a local basis at 0 consisting of subspaces. It can be identified with the pair $(V, \mathcal{F})$ where $\mathcal{F}$ is the filter of open subspaces. The topology is separated if and only if $\bigcap \mathcal{F} = 0$.

Theorem 6 (Lefschetz)

The assignement $LV = (V^*, \textit{weak } *\textit{-topolgy})$, $L(f) = f^*$ defines a full faithful functor $L : \mathrm{Vect}^{\mathrm{op}} \to T\mathrm{Vect}_s$ and its essential image consists of the linearly compact spaces.

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The functor $T\mathrm{Vect}_s(L, -) : T\mathrm{Vect}_s \to [\mathrm{Vect}, \mathrm{Vect}]_{\mathbf{k}}$ factors through the inclusion $\mathrm{Ind}(\mathrm{Vect}^{\mathrm{op}}) \subseteq [\mathrm{Vect}, \mathrm{Vect}]_{\mathbf{k}}$ and the left adjoint of the factor is the left Kan extension of L along the Yoneda embedding $\mathbb{Y} : \mathrm{Vect}^{\mathrm{op}} \to \mathrm{Ind}(\mathrm{Vect}^{\mathrm{op}})$. The associated monad is idempotent and induces an equivalence between the "separated K-spaces" and a reflective proper subcategory of $\Sigma\mathrm{Vect} \subseteq \mathrm{Ind}(\mathrm{Vect}^{\mathrm{op}})$.

Some questions left

- give an example of separated K-space which is not of the form $\mathbf{k}(\Gamma, \mathcal{F})$
- characterize those X which are in $B\Sigma\mathrm{Vect}$
- can we describe the strong linear Kähler differential of the Hahn field $\mathbf{k}((x^{\mathbb{R}}))$?
- $\Sigma\mathrm{Vect}$ is the free part of a torsion theory, is the torsion part $\{X : \forall Y \in \Sigma\mathrm{Vect},\ \mathrm{Nat}(X, Y) = 0\}$ interesting?

Thank you :)

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