Toposes with enough points as categories of étale spaces

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Abstract.

It is well-known that a topology on a set X can be uniquely recovered from a 'convergence' relation between X and ultrafilters on X: in other words, a *spatial locale* can be recovered from its set of points once it is endowed with appropriate extra structure defined in terms of ultrafilters. In this talk, I will present a similar reconstruction result for (Grothendieck) *toposes with enough points*, a categorification of spatial locales: every such topos $\mathcal{E}$ can be recovered up to equivalence from its category of points $\mathrm{pt}(\mathcal{E})$, provided that the latter is endowed with appropriate extra structure involving ultrafilters.

The use of ultrafilters to obtain such a reconstruction theorem dates back to Makkai [2], who introduced *ultracategories* in order to recover a (coherent) first-order theory $\mathbb{T}$, categorically embodied by a pretopos $\mathcal{C}_{\mathbb{T}}$, from its category of models $\mathrm{Mod}(\mathbb{T})$, corresponding to $\mathsf{Pretop}(\mathcal{C}_{\mathbb{T}}, \mathbf{Set})$. At its core, an ultracategory is a category endowed with abstract *ultraproducts*, and the ultracategory structure defined on $\mathrm{Mod}(\mathbb{T})$ by the actual ultraproducts is sufficient to determine $\mathcal{C}_{\mathbb{T}}$ up to equivalence.

Building on this idea, we will introduce *ultraconvergence spaces*. Unlike first-order logic, the category of models of a *geometric* theory need not be closed under the formation of ultraproducts: however, ultraproducts are always defined in the sense of structures for a language, and we can still speak of structure morphisms from a model into such a structure. Therefore, ultraconvergence spaces are intuitively (potentially large) sets endowed with 'formal arrows' into ultraproducts which may not exist, generalizing ultracategories in similar fashion to how multicategories generalize monoidal categories. More concretely, the structure of an ultraconvergence space on a set X is determined by a **Set**-valued relation – that is, a profunctor – between X and *ultrafamilies* in X, i.e. indexed families of elements of X together with an ultrafilter on the indexing set. In particular, the usual convergence relation of a topological space can be seen as a **2**-valued case thereof, so that ultraconvergence spaces encompass all topological spaces.

The same concept was independently introduced for the same aim by Saadia [3] and, with a slightly weaker axiomatization, by Hamad [4]. The main novelty of our approach lies in the introduction of *étale spaces* over an ultraconvergence space, extending the topological case: in these terms, our main theorem can be stated as follows.

Theorem. Let $\mathcal{E}$ be a topos with enough points. Then, $\mathcal{E}$ is equivalent to the category of étale spaces over the ultraconvergence space $\mathrm{pt}(\mathcal{E})$.

This talk is based on joint work with Sam van Gool and Jérémie Marquès [5].

References

[1] M. Makkai, *Stone duality for first order logic*, Adv. in Math. 65 (1987), no. 2, 97–170.

[2] G. Saadia, Extending conceptual completeness via virtual ultracategories, preprint arXiv:2506.23935, 2025.

[3] A. Hamad, Generalised ultracategories and conceptual completeness of geometric logic, preprint arXiv:2507.07922, 2025.

[4] S. van Gool, J. Marquès, and U. Tarantino, Toposes with enough points as categories of étale spaces, preprint arXiv:2508.09604, 2025.