Improper torsion theories

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We consider a variant of the notion of *torsion theory* that is suitable for use in non-pointed contexts and reduces to the usual notion of torsion theory when the categories in question are pointed.

Broadly speaking, equipping a category $\mathcal{A}$ with a torsion theory means picking out two full subcategories $\mathcal{T}$ and $\mathcal{F}$ that are complements of each other in a particular sense, with the "meet" of $\mathcal{T}$ and $\mathcal{F}$ being trivial and the "join" of $\mathcal{T}$ and $\mathcal{F}$ being $\mathcal{A}$, in that each object A of $\mathcal{A}$ admits decomposition into a short exact sequence

$$T \to A \to F\,,$$

with T lying in $\mathcal{T}$ and F lying in $\mathcal{F}$. Different notions of torsion theory arise when one varies what is meant by *trivial* and what it means to be a *short exact sequence*.

Typically, $T \to A \to F$ being a short exact sequence forces $T \to A$ to be monic and $A \to F$ to be epic. Our approach aims to avoid that restriction, since although these morphisms being epic/monic is very helpful in practice, it excludes many examples of interest, such as Artin glueings of toposes.

For instance, given any two categories $\mathcal{T}$ and $\mathcal{F}$, respectively containing a terminal object $\mathbf{1}_{\mathcal{T}}$ and an initial object $\mathbf{0}_{\mathcal{F}}$, one would be tempted to decompose a generic object (T, F) of $\mathcal{T} \times \mathcal{F}$ into a short exact sequence

$$(T, \mathbf{0}_{\mathcal{F}}) \xrightarrow{(\mathsf{id}_T, !)} (T, F) \xrightarrow{(!, \mathsf{id}_F)} (\mathbf{1}_{\mathcal{T}}, F)\,.$$

If it so happens that $\mathbf{0}_{\mathcal{F}} \to F$ is always monic and $T \to \mathbf{1}_{\mathcal{T}}$ is always epic, then these decompositions indeed yield a *pretorsion theory* in the sense of [1], but even without that assumption this decomposition is no less sensible.

The word *improper* signifies the lack of the monic/epic assumption, similarly to how in an improper orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$, the class $\mathcal{E}$ is not required to consist of epics nor the class $\mathcal{M}$ of monics. Indeed, in the category $\mathsf{Arr}(\mathcal{C})$ of arrows of $\mathcal{C}$, the short exact sequences

$$\begin{array}{ccccccc} A & \xrightarrow{1} & A & \xrightarrow{e} & E & & \\ & {\scriptstyle e}\searrow & & {\scriptstyle f}\searrow & & {\scriptstyle m}\searrow & \\ & & E & \xrightarrow[m]{} & B & \xrightarrow[1]{} & B \end{array}$$

induced by the $(\mathcal{E}, \mathcal{M})$-factorizations only give rise to a pretorsion theory when the factorization system is proper.

References

[1] A. Facchini, C. Finocchiaro, M. Gran, Pretorsion theories in general categories. *J. Pure Appl. Algebra* 225 (2021), no. 2, Paper No. 106503, 21 pp.

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