Improper torsion theories

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We consider a variant of the notion of *torsion theory* that is suitable for use in non-pointed contexts and reduces to the usual notion of torsion theory when the categories in question are pointed.

Broadly speaking, equipping a category \mathcal{A} with a torsion theory means picking out two full subcategories \mathcal{T} and \mathcal{F} that are complements of each other in a particular sense, with the "meet" of \mathcal{T} and \mathcal{F} being trivial and the "join" of \mathcal{T} and \mathcal{F} being \mathcal{A} , in that each object A of \mathcal{A} admits decomposition into a short exact sequence

$$T \to A \to F$$
,

with T lying in \mathcal{T} and F lying in \mathcal{F} . Different notions of torsion theory arise when one varies what is meant by trivial and what it means to be a short exact sequence.

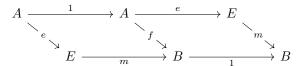
Typically, $T \to A \to F$ being a short exact sequence forces $T \to A$ to be monic and $A \to F$ to be epic. Our approach aims to avoid that restriction, since although these morphisms being epic/monic is very helpful in practice, it excludes many examples of interest, such as Artin glueings of toposes.

For instance, given any two categories \mathcal{T} and \mathcal{F} , respectively containing a terminal object $\mathbf{1}_{\mathcal{T}}$ and an initial object $\mathbf{0}_{\mathcal{F}}$, one would be tempted to decompose a generic object (T, F) of $\mathcal{T} \times \mathcal{F}$ into a short exact sequence

$$(T,\mathbf{0}_{\mathcal{F}}) \xrightarrow{(\mathsf{id}_T,!)} (T,F) \xrightarrow{(!,\mathsf{id}_F)} (\mathbf{1}_{\mathcal{T}},F)\,.$$

If it so happens that $\mathbf{0}_{\mathcal{F}} \to F$ is always monic and $T \to \mathbf{1}_{\mathcal{T}}$ is always epic, then these decompositions indeed yield a *pretorsion theory* in the sense of [1], but even without that assumption this decomposition is no less sensible.

The word *improper* signifies the lack of the monic/epic assumption, similarly to how in an improper orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ on \mathcal{C} , the class \mathcal{E} is not required to consist of epics nor the class \mathcal{M} of monics. Indeed, in the category $\mathsf{Arr}(\mathcal{C})$ of arrows of \mathcal{C} , the short exact sequences



induced by the $(\mathcal{E}, \mathcal{M})$ -factorizations only give rise to a pretorsion theory when the factorization system is proper.

References

[1] A. Facchini, C. Finocchiaro, M. Gran, Pretorsion theories in general categories. *J. Pure Appl. Algebra* 225 (2021), no. 2, Paper No. 106503, 21 pp.

^{*}Joint work with Fosco Loregian, Graham Manuell and Nelson Martins-Ferreira.