Unbiased multicategories, concretely

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The languages of double categories and of fibrations provide a natural framework for unbiased notions of multicategories; here, "unbiased" refers to the fact that we deal with families (rather than sequences) of objects and arrows. In fact, symmetric multicategories are simply sum-preserving double discrete fibrations $\mathbb{M} \to \mathbb{P}\mathrm{b}$ to the double category of pullback squares in finite sets.

This approach turns out to be rather effective and opens up new perspectives. In particular, it allows for a base sensitive study of multicategories and renders transparent the links with Joyal's species. For instance, plain multicategories are sum-preserving double discrete fibrations $\mathbb{M} \to \mathbb{T}$, where $\mathbb{T}$ is (the double categorical form of) the species of total orders, with its natural composition.

We can view the base as a theory, so that symmetric (respectively, plain) multicategories are the model for $\mathbb{P}\mathrm{b}$ (respectively, for $\mathbb{T}$); among them, when the loose part of the functor is an opfibration, there are the representable ones, corresponding to (symmetric) monoidal multicategories, and, when the loose part is a discrete opfibration, the strict ones, corresponding (commutative) monoids. Every domain $\mathbb{M}$ of a $\mathbb{P}$-model $\mathbb{M} \to \mathbb{P}$ is itself a theory. In particular, any multicategory $\mathbb{M} \to \mathbb{P}\mathrm{b}$ has its own models, the strict ones being the usual algebras for the multicategory.

Cartesian multicategories fit nicely in this setting; a cartesian structure on $\mathbb{M} \to \mathbb{P}\mathrm{b}$ is a way to evaluate spans formed by a tight arrow f and a loose arrow α in $\mathbb{M}$, giving the "sum" $\sum_f \alpha$ of α along f. The key condition is that the sum of a composition of spans is the same as the composition of their sums, where spans in $\mathbb{M}$ are composed in the usual way, except that we use $\mathbb{M}$-cells in place of pullbacks:

$$\begin{tikzcd}
 & & W \arrow[dl, "h"'] \arrow[dr, "\gamma", "|" marking] & & \\
 & U \arrow[dl, "f"'] \arrow[dr, "\alpha", "|" marking] \arrow[rr, phantom, "@"] & & V \arrow[dl, "g"'] \arrow[dr, "\beta", "|" marking] & \\
X \arrow[rr, "\sum_f \alpha"', "|" marking] & & Y \arrow[rr, "\sum_g \beta"', "|" marking] & & Z
\end{tikzcd}$$

$$\textstyle\sum_g \beta \sum_f \alpha = \sum_{fh} \beta\gamma$$

Since cells in $\mathbb{M}$ (like @) are defined by reindexing, this can be seen as a kind of generalized distributive law, holding in any cartesian multicategory. If M is a monoid and $\mathbb{M}$ is the symmetric one-object multicategory given by families of elements of M, then cartesian structures on $\mathbb{M}$ correspond to rig structures on M and it becomes the standard distributive law.

In fact, we define cartesian multicategories as the algebras for the monad which takes a symmetric multicategory $\mathbb{M}$ to the multicategory $\mathbb{M}^{\text{cart}}$ whose loose arrows are spans (actually, "enhanced" spans) in $\mathbb{M}$. The above distributive law corresponds to the functoriality (on loose arrows) of the algebra structure map $\sum : \mathbb{M}^{\text{cart}} \to \mathbb{M}$.

In this talk we present these ideas in a concrete way, through a few simple instances, rather than aiming to great generality.

References

[1] C. Pisani, Unbiased multicategory theory, TAC (2025).