

# Unbiased multicategories, concretely

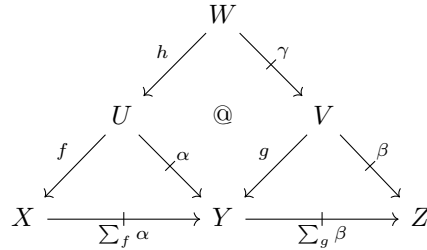
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The languages of double categories and of fibrations provide a natural framework for unbiased notions of multicategories; here, “unbiased” refers to the fact that we deal with families (rather than sequences) of objects and arrows. In fact, symmetric multicategories are simply sum-preserving double discrete fibrations  $\mathbb{M} \rightarrow \mathbb{Pb}$  to the double category of pullback squares in finite sets.

This approach turns out to be rather effective and opens up new perspectives. In particular, it allows for a base sensitive study of multicategories and renders transparent the links with Joyal’s species. For instance, plain multicategories are sum-preserving double discrete fibrations  $\mathbb{M} \rightarrow \mathbb{T}$ , where  $\mathbb{T}$  is (the double categorical form of) the species of total orders, with its natural composition.

We can view the base as a theory, so that symmetric (respectively, plain) multicategories are the model for  $\mathbb{Pb}$  (respectively, for  $\mathbb{T}$ ); among them, when the loose part of the functor is an opfibration, there are the representable ones, corresponding to (symmetric) monoidal multicategories, and, when the loose part is a discrete opfibration, the strict ones, corresponding (commutative) monoids. Every domain  $\mathbb{M}$  of a  $\mathbb{P}$ -model  $\mathbb{M} \rightarrow \mathbb{P}$  is itself a theory. In particular, any multicategory  $\mathbb{M} \rightarrow \mathbb{Pb}$  has its own models, the strict ones being the usual algebras for the multicategory.

Cartesian multicategories fit nicely in this setting; a cartesian structure on  $\mathbb{M} \rightarrow \mathbb{Pb}$  is a way to evaluate spans formed by a tight arrow  $f$  and a loose arrow  $\alpha$  in  $\mathbb{M}$ , giving the “sum”  $\sum_f \alpha$  of  $\alpha$  along  $f$ . The key condition is that the sum of a composition of spans is the same as the composition of their sums, where spans in  $\mathbb{M}$  are composed in the usual way, except that we use  $\mathbb{M}$ -cells in place of pullbacks:



$$\sum_g \beta \sum_f \alpha = \sum_{fh} \beta \gamma$$

Since cells in  $\mathbb{M}$  (like  $@$ ) are defined by reindexing, this can be seen as a kind of generalized distributive law, holding in any cartesian multicategory. If  $M$  is a monoid and  $\mathbb{M}$  is the symmetric one-object multicategory given by families of elements of  $M$ , then cartesian structures on  $\mathbb{M}$  correspond to rig structures on  $M$  and it becomes the standard distributive law.

In fact, we define cartesian multicategories as the algebras for the monad which takes a symmetric multicategory  $\mathbb{M}$  to the multicategory  $\mathbb{M}^{\text{cart}}$  whose loose arrows are spans (actually, “enhanced” spans) in  $\mathbb{M}$ . The above distributive law corresponds to the functoriality (on loose arrows) of the algebra structure map  $\sum : \mathbb{M}^{\text{cart}} \rightarrow \mathbb{M}$ .

In this talk we present these ideas in a concrete way, through a few simple instances, rather than aiming to great generality.

## References

- [1] C. Pisani, Unbiased multicategory theory, TAC (2025).