A topos for extended Weihrauch degrees

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In recent years, there have been some works starting to approach computability-like notions from a categorical perspective (see e.g. [5, 4, 6, 7]). In [1], Bauer introduced an abstract notion of reducibility between predicates, called instance reducibility, which commonly appears in reverse constructive mathematics. In a relative realizability topos, the instance degrees correspond to a generalization of (realizer-based) Weihrauch reducibility, called *extended Weihrauch degrees*, and the "classical" Weihrauch degrees [2] correspond precisely to the ¬¬-dense modest instance degrees in Kleene-Vesley realizability. Upon closer inspection, it is not hard to check that realizer-based Weihrauch reducibility is a particular case of Bauer's notion.

The main goal of this talk is to show how one can define a topos for extended Weihrauch degrees, providing a suitable universe for studying this reducibility categorically. Then, we take advantage from this categorical presentation, and we establish the precise connection between extended Weihrauch degrees and realizability.

The main tools we adopt to construct such a topos are: (i) the *tripos-to-topos construction* [3], which produces a topos from a given tripos (that is, a particular kind of Lawvere hyperdoctrine which has enough structure to deal with higher-order logic properly), and (ii) the *(full) existential completion*, a construction that freely adds left adjoints along all the morphisms of the base of a given doctrine.

References

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