An Equivariant Model for Cubical Type Theory

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Homotopy type theory (HoTT) is a formal system to reason about homotopy theory in a synthetic way in that the basic objects are spaces up to homotopy rather than discrete sets[Voe06; Awo10; Rij22; Uni13]. As a type theory, it is amenable to implementation as a programming language. However, due to the presence of Voevodsky's univalence axiom, HoTT in its standard variant is not fully computational. To remedy this problem, Coquand et al. introduced cubical type theory (CTT) [BCH14; CCHM16; OP18; ABCHFL21] in which univalence can indeed be derived as a theorem.

The standard interpretation of HoTT takes place in Kan simplicial sets [HS97; AW09; KL18; Str14] which provide a model of the homotopy theory of spaces up-to-homotopy. For cubical type theory, it is natural to ask if the ensuing models in cubical sets also present the standard homotopy theory of spaces. CTT can be interpreted with respect to different cube categories [BM17; CMS20; Awo18; Awo23], many of which are either unknown or known to not give rise to the standard homotopy of spaces due to unpublished arguments by Buchholtz and Sattler, see also [SW21; HR22; KV20].

Intuitively, this failure is caused by the presence of additional symmetries (compared to the simplicial model) that make the shape arising from folding a square along its diagonal nontrivial in the cubical setting though it is trivial as a topological space. The first model that was developed to have this property—modelling CTT while presenting classical homotopy theory—are *equivariant* cubical spaces [ACCRS24]. In technical terms, the main idea to remedy the discrepancy between cubical and classical homotopy theory is to make the fibrations of the model structure cohere with permutations of the cube dimensions. As a consequence, the point inclusions into these folded cubes become trivial cofibrations.

In the talk, I will present an account to the model of [ACCRS24] worked out in my masters thesis [Fri24].

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