On the lack of colimits in various categories of BAOs and Heyting algebras

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The varieties of Boolean algebras and of Heyting algebras are some of the most studied classes of algebras in algebraic logic. Since every prevariety (i.e., a class of algebras closed under isomorphisms, subalgebras, and products) viewed as a category is complete and cocomplete (see, e.g., [11, Thm. IV.2.1.3 and IV.2.2.3]), the categories of Boolean algebras with Boolean homomorphisms and of Heyting algebras with Heyting homomorphisms are complete and cocomplete. The situation changes when we restrict our attention to the categories **CBA** of complete Boolean algebras with complete Boolean homomorphisms and **CHA** of complete Heyting algebras with complete Heyting homomorphisms. Indeed, while free complete Boolean algebras on finitely many generators exist, it is a classic result from the 1960s that the free complete Boolean algebra on countably many generators does not exist [6, 8]. The one-generated free complete Heyting algebra exists (see, e.g., [9, Sec. I.4.11]), but De Jongh [5] proved that already the two-generated free complete Heyting algebra does not exist (see also [3]). It follows that both **CBA** and **CHA** lack some coproducts, and hence neither is cocomplete.

In this talk we will discuss similar results on the lack of colimits in various categories of Boolean algebras, Heyting algebras, and Boolean algebras with operators (BAOs). In particular, we will see that the categories of complete Boolean algebras with Boolean homomorphisms, the category of complete Heyting algebras with Heyting homomorphisms, and the category of (complete) Heyting algebras with bounded lattice homomorphisms are not cocomplete. A similar phenomenon occurs in categories of BAOs with stable morphisms, which are Boolean homomorphisms $f\colon A \to B$ satisfying $\Diamond f(a) \le f(\Diamond a)$ for each $a \in A$. Stable morphisms play an important role in modal logic [7, 2] and in the *Algebra of Topology* of McKinsey and Tarski [10, 4]. We will see that the category of McKinsey-Tarski algebras (i.e., complete closure algebras) with (complete) stable morphisms is cocomplete. These results generalize to several categories of complete BAOs with (complete) stable morphisms, as well as to categories of BAOs with stable morphisms. The lack of colimits implies that none of these categories is equivalent to a prevariety, let alone to a variety

Our main tool is duality theory: Stone duality for Boolean algebras, Esakia and Priestley dualities for Heyting algebras and bounded distributive lattices, and Jónsson-Tarski duality for BAOs. These dualities reduce the problem to showing that the corresponding dual categories are not complete. This we do by proving that in these dual categories either products or equalizers do not exist.

The results presented in this talk are collected in [1].

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