The doctrinal Herbrand's theorem and its Stone dual

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Herbrand's theorem (in its weak form) says that if $\exists x\, \phi(x)$ is provable (with $\phi(x)$ quantifier-free), then some finite disjunction $\phi(t_1) \vee \cdots \vee \phi(t_n)$ of its ground instances is already provable.

In [AG], we obtained Herbrand's theorem within the formalism of first-order Boolean doctrines. We recall that *first-order Boolean doctrines* are the analogue, for classical first-order logic, of Boolean algebras for classical propositional logic: given a first-order theory $\mathcal{T}$, one considers formulas (indexed by their contexts of free variables) modulo $\mathcal{T}$-provable equivalence, and obtains a first-order Boolean doctrine. Formally, a *first-order Boolean doctrine* over a category C with finite products is a functor $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ in which the reindexings along the product projections have right and left adjoints satisfying the Beck-Chevalley condition.

Recall now that a (not necessarily first-order) *Boolean doctrine* over a category C with finite products is simply a functor $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$; a Boolean doctrine models the class of quantifier-free formulas modulo a universal first-order theory, indexed by their contexts. Any Boolean doctrine over a small category admits a quantifier completion $\mathbf{P} \hookrightarrow \mathbf{P}^{\forall\exists}$, i.e., a universal morphism into a *first-order* Boolean doctrine, which freely adds first-order quantifiers to $\mathbf{P}$. Herbrand's theorem gives an explicit description of the fragment of $\mathbf{P}^{\forall\exists}$ consisting of formulas of quantifier-alternation depth at most 1.

In this talk, I will present joint work in progress with Francesca Guffanti, in which we develop the Stone dual of this description. Composing a first-order Boolean doctrine $\mathbf{P}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{BA}$ with Stone duality $\mathsf{BA}^{\mathrm{op}} \cong \mathsf{Stone}$ yields Joyal's notion of polyadic space, namely a functor $\mathbf{E}\colon \mathsf{C} \to \mathsf{Stone}$ in which product projections are sent to open maps, and certain squares are epi-pullbacks. Just as a Stone space captures the space of models of a classical propositional theory, a polyadic space captures the space of models of a classical first-order theory, up to elementary equivalence.

For the Stone dual of Herbrand's theorem, we start from a Stone-valued functor $\mathbf{E}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Stone}$, representing the space of models of a universal theory modulo agreement on quantifier-free formulas. We then give an explicit description—entirely in terms of $\mathbf{E}$—of the Stone-valued functor capturing the space of models of the same theory modulo agreement on formulas of quantifier-alternation depth at most 1. In other words, we identify the first nontrivial layer of the quantifier completion on the Stone/polyadic side, thus obtaining the Stone dual of the doctrinal Herbrand theorem.

References

[AG] M. Abbadini and F. Guffanti. Freely adding one layer of quantifiers to a Boolean doctrine. Preprint at https://arxiv.org/abs/2410.16328.