Higher dimensional semantics of propositional dependent type theories

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In the study of the semantics of dependent type theory, we typically encounter two distinct, but related, approaches: *syntactical* and *categorical*. The syntactical approach directly mirrors the structure of the syntax of the theory, via a choice function that defines how judgements in the conclusions of inference rules are interpreted based on the interpretations assigned to the premises (see e.g. [Hof97] and [Str91]). On the other hand, the categorical approach adds structure and property to the model: structure and property that allows to recover a choice function analogous to that provided in the syntactical approach. For instance, assuming that the notion of equality of a given theory is *extensional*, the inference rules of the dependent sum type constructor correspond to the requirement that the family of display maps in a model is closed under composition up to isomorphism: this categorical property encodes into the model the syntax of dependent sums (see e.g. [HS98] and [Jac99]).

However, in case we drop the extensionality of the identity types, such a simple and clear characterisation of the type constructors is harder to find, and hence a (1-dimensional) categorical presentation of the semantics of such a theory is not completely satisfactory. Given this challenge of encoding *intensional* theories into 1-dimensional categorical terms, a challenge that persists even for *propositional* theories — i.e. *very intensional* theories, also known in the literature as *axiomatic*, *weak*, *objective* theories — in this talk we adopt Garner's perspective [Gar09] to study the semantics of propositional theories from a *higher* categorical point of view, specifically a 2-categorical one.

A propositional theory [AGS17, CD13, vdB18, vdB23, BW19, Boc22, OS24] is a dependent type theory whose computation rules consist of propositional equalities, rather than the definitional equalities that normally characterise the reductions and the expansions in formal systems like Martin-Löf type theory or the calculus of constructions. Starting from the syntax of a propositional theory, we prove that its type constructors can be encoded into natural 2-dimensional category theoretic data. We use these data to show that the semantics of propositional theories of dependent types admits a presentation via 2-categorical models called *display map 2-categories*. In other words, we show that display map 2-categories with such data are sufficient to reconstruct the semantic counterpart of propositional theories as in the syntactical approach, particularly inducing appropriate display map categories — i.e ordinary models [Tay99, Jac99]. It turns out that the 2-categorical requirements identified by Garner for interpreting an intensional theory, in the propositional case are relaxed. Therefore, we obtain a notion of semantics for propositional theories that generalises Garner's one for intensional ones. We compare the class of models according to this notion of semantics with the class of those derived from the usual notion of semantics.

References

[AGS17] S. Awodey, N. Gambino, and K. Sojakova. Homotopy-initial algebras in type theory. *J. ACM*, 63(6):Art. 51, 45, 2017.

[Boc22] R. Bocquet. Strictification of weakly stable type-theoretic structures using generic contexts. In *27th International Conference on Types for Proofs and Programs*, volume 239 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 3, 23. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2022.

[BW19] S. Boulier and T. Winterhalter. Weak type theory is rather strong. *30th International Conference on Types for Proofs and Programs*, 2019. `https://www.ii.uib.no/~bezem/abstracts/TYPES_2019_paper_18`.

[CD13] T. Coquand and N. A. Danielsson. Isomorphism is equality. *Indag. Math. (N.S.)*, 24(4):1105–1120, 2013.

[Gar09] R. Garner. Two-dimensional models of type theory. *Math. Structures Comput. Sci.*, 19(4):687–736, 2009.

[Hof97] M. Hofmann. *Syntax and semantics of dependent types*, pages 13–54. Springer, London, 1997.

[HS98] M. Hofmann and T. Streicher. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory (Venice, 1995)*, volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, 1998.

[Jac99] B. Jacobs. *Categorical logic and type theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1999.

[OS24] D. Otten and M. Spadetto. Models for axiomatic type theory. *Workshop on Homotopy Type Theory & Univalent Foundations*, 2024. `https://hott-uf.github.io/2024/abstracts/HoTTUF_2024_paper_14.pdf`.

[Str91] T. Streicher. *Semantics of type theory — Correctness, completeness and independence results*. Progress in Theoretical Computer Science. Birkhäuser Boston, Inc., Boston, MA, 1991.

[Tay99] P. Taylor. *Practical foundations of mathematics*, volume 59 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.

[vdB18] B. van den Berg. Path categories and propositional identity types. *ACM Trans. Comput. Log.*, 19(2):Art. 15, 32, 2018.

[vdB23] B. van den Berg. Towards homotopy canonicity for propositional type theory. *Dutch Categories And Types Seminar*, 2023. `https://dutchcats.github.io/Weak_type_theories/slides_van_den_berg.pdf`.