Abstract, 5th ItaCa Workshop

A Gelfand duality adjunction for measurable spaces

Tobias Fritz and *Antonio Lorenzin*

Universität Innsbruck, Institut für Mathematik, Technikerstraße 25, 6020 Innsbruck, Austria

Gelfand duality is a well-known theorem establishing a contravariant equivalence between compact Hausdorff spaces and commutative C^*-algebras.[1] In particular, noncommutative C^*-algebras can be thought of as noncommutative spaces with the directions of arrows reversed. However, C^*-algebras also play an important role in quantum mechanics as algebras of observables.

The inherent probabilistic aspects of quantum mechanics makes the theory of C^*-algebras of interest also from a probabilistic perspective, and there are intriguing similarities. In particular, there is a probabilistic Gelfand duality [FJ15], which shows that completely positive maps between commutative C^*-algebras correspond to *continuous* Markov kernels between their associated spaces. However, this continuity requirement is unsatisfactory from a probabilistic standpoint, since Markov kernels that arise in practice (e.g. as regular conditional probabilities) often fail to be continuous. For this reason, we doubt that the category of C^*-algebras really is the best setting for quantum probability theory.

Related issues arise in the recent theory of Markov categories, which allows us to study probability theory from a synthetic point of view by abstracting away from measure-theoretical details. There, it is an open problem to find a well-behaved Markov category for measure-theoretic probability; the usual choice of $\mathsf{BorelStoch}$ — whose objects are *standard Borel spaces* — seems deficient insofar as it does not admit measurable spaces "larger" than $\mathbb{R}$.

Thus the goal of the present work is to investigate alternative categories for both the classical and the quantum setting of probability theory. In the classical case, the recent work by Jamneshan and Tao [JT23] has very similar goals. As they argue, a good choice for the objects is given by ***Baire measurable spaces***, which extend standard Borel spaces out of the countable setting in a nice way; for instance, conditional probabilities still exist [Fre03, Corollary 452N]. It is of interest to understand whether C^*-algebras can be used to understand these spaces from an algebraic perspective, especially as many proofs in measure theory are secretly algebraic in nature.

In the present paper, we therefore consider generalizations of Gelfand duality in the context of measurable spaces, both for the deterministic side (measurable functions) and for the probabilistic side (Markov kernels). Let us first focus on the deterministic side. Our main theorem, the ***measurable Gelfand duality***, states the existence of a contravariant idempotent adjunction between measurable spaces and commutative ***monotone σ-complete C^*-algebras***, which are C^*-algebras satisfying a monotone convergence theorem.

This adjunction nicely restricts to an equivalence between Baire measurable spaces and commutative ***Pedersen–Baire envelopes***, whose construction is analogous to that of enveloping von Neumann algebras. Moreover, standard Borel spaces correspond to the Pedersen–Baire envelopes of commutative *separable* C^*-algebras. This equivalence is also preserved in the probabilistic case, where Markov kernels are associated with completely positive maps satisfying a notion of σ-additivity.

In order to ensure that this equivalence respects the Markov category structure, one first needs to understand what tensor product should be adopted in the algebraic setting. Already for commutative monotone σ-complete C^*-algebras, this poses nontrivial problems, as there are two distinct and relevant tensor products. Our main result in this direction shows that these actually coincide for commutative Pedersen–Baire envelopes, therefore ensuring a meaningful equivalence of Markov categories between Baire measurable spaces and commutative Pedersen–Baire envelopes.

References

[FJ15] Robert W. J. Furber and Bart P. F. Jacobs. From Kleisli categories to commutative C*-algebras: Probabilistic Gelfand duality. *Logical Methods in Computer Science*, 11, 2015. arXiv:1303.1115.

[Fre03] David H. Fremlin. *Measure theory. Vol. 4.* Torres Fremlin, Colchester, 2003. Topological Measure Spaces. Available at essex.ac.uk/maths/people/fremlin/mt.htm.

[JT23] Asgar Jamneshan and Terence Tao. Foundational aspects of uncountable measure theory: Gelfand duality, Riesz representation, canonical models, and canonical disintegration. *Fundam. Math.*, 261(1):1–98, 2023.

[1] For our purposes, C^*-algebras and their homomomorphisms are always *unital*.