Directed equality with dinaturality*

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Equality in Martin-Löf type theory is inherently symmetric [7]: this is what allows for types to be interpreted as sets, groupoids [8], and ∞-groupoids [4], where points of a type correspond to objects and equality is precisely interpreted by morphisms which are always invertible.

A natural question follows: can there be a variant of Martin-Löf type theory which enables types to be interpreted as *categories*, where morphisms need not be invertible? Such a system should take the name of *directed type theory* [13, 16, 1, 6, 10, 2] (DTT), where the directed aspect comes from a non-symmetric interpretation of "equality", which now has a source and a target in the same way that morphisms do in a category. A common feature of current semantic approaches to directed type theory is to resort back to the maximal sub*groupoid* $\mathbb{C}^{\mathsf{core}}$ of categories [16, 2] in order to use the same variable with different variances $\mathbb{C}$ and $\mathbb{C}^{\mathsf{op}}$; this is needed to validate introduction (refl) and elimination rules (J-rule) for directed equality.

Dinaturality. In this work, we describe how *dinatural transformations* [5] allow us to semantically validate both of the above rules exactly in the style of MLTT, where directed equality is interpreted by hom-functors and types by (1-)categories. The syntactic requirement that needs to be imposed on the J-rule is that, given a directed equality in context $\hom_{\mathbb{C}}(x, y)$ for $x : \mathbb{C}^{\mathsf{op}}, y : \mathbb{C}$ both x and y must appear only positively (i.e., with the same variance) in the conclusion and only negatively (i.e., with the opposite variance) in the context. These rules allow us to syntactically recover the same definitions about symmetric equality that one expects in standard Martin-Löf type theory, *except for symmetry*: e.g., transitivity of directed equality (composition in a category), congruences of terms along directed equalities (the action of a functor on morphisms), transport along directed equalities (i.e., the coYoneda lemma).

(Co)ends as quantifiers. Moreover, we show how dinaturality allows us to more precisely view (co)ends [14] as the "directed quantifiers" of DTT, which we present in a correspondence reminiscent of the quantifiers-as-adjoint paradigm of Lawvere [11]. We do not provide an account of these rules using categorical semantics precisely because dinaturals do not compose in general [5]; despite this lack of general composition, the rules for directed equality and coends-as-quantifiers can be used to give concise proofs of theorems in category theory using a distinctly *logical* flavour via a series of isomorphisms: e.g., the (co)Yoneda lemma, Kan extensions computed via (co)ends are adjoints, presheaves form a closed category, hom preserves (co)limits, and Fubini, which easily follow by modularly using the logical rules of each connective. This constitutes a concrete step towards formally understanding the so-called "coend calculus" [14].

Directed equality as adjoint. Symmetric equality has a well-known characterization as left adjoint to contraction functors [9, 3.4.1], [12]. We present a similar characterization for directed equality in terms of a *left relative adjunction* [17, 3] which views hom-functors $\hom : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$ as *relative left adjoints* to certain "contraction-like" functors between *para*categories of dinatural transformations (i.e. where composition is partially defined) which, intuitively, join two *natural* variables into a single *dinatural* one. The relative adjunction is semantically justified by the J-rule, and the *relativeness* captures its syntactic restrictions. This suggests a tentative answer to a problem first posed by Lawvere on the precise role played by hom for the presheaf hyperdoctrine [12, p.11], where the Beck-Chevalley and Frobenius conditions fail for equality [15]. In the posetal case, the above becomes a genuine left relative adjunction: we highlight our current progress towards a *directed logic* of posets and an axiomatization of the notion of *directed doctrine* which adequately captures variances and (di)naturality.

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