Presheaves as Kripke frames

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Kripke frames (directed graphs) are the main ingredient of the most popular semantics for modal logics [3]. Together with *p-morphisms* (functions f satisfying $f(v^{\uparrow}) = f(v)^{\uparrow}$, for each vertex v), they form the category $\mathbf{KFr}$. $\mathbf{KFr}_{lf}$, the class of *locally finite* Kripke frames (for each vertex v, the set of endpoints of directed paths starting from v is finite), has some nice categorical properties. It is cocomplete and complete, being the Ind-completion (see [4]) of $\mathbf{KFr}_f$, and the forgetful functor $\mathbf{KFr}_{lf} \longrightarrow \mathbf{Set}$ is co-monadic. All these properties are due to semantic reasons, namely the closure of $\mathbf{KFr}_{lf}$ under the standard truth-preserving operations of taking generated subframes, p-morphic images and disjoint unions. As a consequence, any subclass $\mathcal{C} \subseteq \mathbf{KFr}_{lf}$ closed under the aforementioned operations enjoys the same categorical properties (see [2] for a complete overview).

It is natural to ask, assuming that all the Kripke frames in $\mathcal{C}$ are transitive (so that $\mathcal{C}$ uniquely corresponds to some normal modal logic containing $\mathbf{K4}$ and with the finite model property), whether $\mathcal{C}$ is a regular category or not. It turns out that $\mathcal{C}$ is regular if and only if $\mathcal{C}_f$ has the co-amalgamation property. In the case of preorders, with techniques similar to those used in [5], it is possible to exclude regularity for every but 49 cases.

Barr exactness can be characterized, at least in the cases of preorders and of strict preorders, by describing products in $\mathcal{C}$ by means of the *universal model construction* (well known in the modal logic literature [1]). Moreover, exactness is sufficient for $\mathcal{C}$ to be a Grothendieck topos (Giraud's characterization).

Vice versa, there is a canonical way to see presheaves as Kripke frames. Given a small category $\mathcal{D}$, and a presheaf F in $\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$, we can consider the slice category $\mathcal{D}/F$. The objects of $\mathcal{D}/F$ are natural transformations $x\colon D \longrightarrow F$, with D in $\mathcal{D}$ (identifying D in $\mathcal{D}$ with the representable presheaf $\mathrm{Hom}_{\mathcal{D}}(\_, D)$); morphisms $y \longrightarrow x$ in $\mathcal{D}/F$ are given by arrows $a\colon E \longrightarrow D$ in $\mathcal{D}$ such that $y = x \circ a$. We can then consider the preorder $K_{\mathcal{D}}(F)$, with underlying set $\mathcal{D}/F$, obtained by setting $x \leq y$ iff $y = x \circ a$ for some $a\colon E \longrightarrow D$ in $\mathcal{D}$. A natural transformation $\alpha\colon F \longrightarrow F'$ in $\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$ induces a functor between the corresponding slices, by composition. As a function $K_{\mathcal{D}}(F) \longrightarrow K_{\mathcal{D}}(F')$, it is a p-morphism. Putting everything together, $K_{\mathcal{D}}$ defines a faithful functor from $\mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$ to $\mathbf{KFr}$. Its image, contained in the class of preorders, is closed under generated subframes and disjoint unions. We will characterize those $\mathcal{D}$ for which $K_{\mathcal{D}}$ is an equivalence with a class $\mathcal{C}$ of locally finite Kripke frames having all the closure properties we want.

References

[1] Fabio Bellissima, *An effective representation for finitely generated free interior algebras*, Algebra Universalis, 20(3):302-317, 1985.

[2] Matteo De Berardinis and Silvio Ghilardi, *Profiniteness, monadicity and universal models in modal logic*, Annals of Pure and Applied Logic, Volume 175, Issue 7, 2024.

[3] Alexander Chagrov and Michael Zakharyaschev, *Modal logic*, volume 35 of Oxford Logic Guides, The Clarendon Press, Oxford University Press, New York, 1997, Oxford Science Publications.

[4] Peter T. Johnstone, *Stone spaces*, volume 3 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1982.

[5] Larisa L. Maksimova, *Interpolation theorems in modal logics and amalgamable varieties of topological Boolean algebras*, Algebra and Logic, 18:348-370, 1979.