Doctrines of algebras as extensional quotient completions *

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Relational doctrines [3, 4] provide a functorial description, in the spirit of Lawvere's hyperdoctrines [5, 6], of a core fragment of the calculus of relations [1, 10, 11]. In this context, one can define equivalence relations and their quotients as well as a universal construction, dubbed *extensional quotient completion*, which freely adds (extensional) quotients to any relational doctrine. This construction generalizes both the elementary quotient completion of existential elementary doctrines [8, 9] and the exact completion of categories with weak finite limits [2].

An important result about the exact completion of a category with weak finite limits states that the category of algebras for a monad on an exact category $\mathcal{C}$ is the exact completion of its Kleisli category, provided that $\mathcal{C}$ satisfies a form of the Axiom of Choice [12]. In this talk, we will show how this result extends to the extensional quotient completion of relational doctrines. To achieve this, we will first characterize doctrines obtained by the extensional quotient completion as those haveing quotients and *enough projectives*, extending similar results about the elementary quotient completion [7] and the exact completion [2]. Then, we will describe the Eilenberg-Moore construction for monads on relational doctrines, obtaining the following theorem.

Theorem 1 *Let $R : (\mathcal{C} \times \mathcal{C})^{\mathrm{op}} \to \mathcal{P}os$ be an extensional relational doctrine with quotients. Then, the following are equivalent*

1. *Every R-surjective arrow in $\mathcal{C}$ splits.*
2. *For every monad $\mathbb{T} = \langle T, \eta, \mu \rangle$ on R, the Eilenberg-Moore doctrine $R^{\mathbb{T}}$ is balanced, extensional, has quotients and free algebras are $R^{\mathbb{T}}$-projective.*

*This is joint work with Fabio Pasquali

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