On categorical notions of context

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(joint work with David Jaz Myers)

We introduce *contextads* and the $\mathfrak{Ctx}$ *construction*, unifying various categorical structures and constructions used to deal with context and contextful maps in category theory and applications—namely actegories and their Para construction [CGHR22], adequate triples and their Span construction [HHLN23], comonads and their Kleisli construction.

The talk will follow the natural narrative bringing about this unification and leading to the definition of contextads as Lack–Street (pseudoco)wreaths [LS02], although one dimension higher.

Our construction of the tricategory of contextads and the associated trifunctor $\mathfrak{Ctx}$: $\mathfrak{Cxd} \to \mathfrak{PsCat}$ is parametric on a kind of 2-category with display maps $(\mathbb{K}, \mathcal{D})$. We apply work of Miranda [Mir24] who defined the Kleisli completion of a tricategory and showed how to recover the tricategory of pseudocategories, pseudofunctors, loose natural transformations and modifications from a tricategory of spans. Thus starting from $(\mathbb{K}, \mathcal{D})$, we look at the Kleisli completion of the associated tricategory of left-displayed spans. Restricting to left-fibrant spans, we see how pseudomonads in this tricategory are contextads, and the associated wreath product is the context construction.

By working parametric on $(\mathbb{K}, \mathcal{D})$ we can easily internalize 2-algebraic structure. We prove that if a contextad is *colaxly T-structured*, meaning equipped with a T-algebra structure for a 2-monad T on the ambient 2-category $\mathbb{K}$, then so is the double category of contextful maps obtained from it. This result generalizes a plethora of (sometimes only folkloric) results about structured (double) categories of parametric arrows, spans, and Kleisli maps.

The talk follows [CM24].

References

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