TORSION THEORIES AND FACTORIZATION SYSTEMS IN A NON-POINTED CONTEXT

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Let $\mathcal{C}$ be a pointed category. A *torsion theory* in $\mathcal{C}$ is defined as a pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories such that:

i) $\mathrm{Hom}(T, F) = \{0\}$ for every $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
ii) for every object $A \in \mathcal{C}$, there exists an exact sequence $T(A) \to A \to F(A)$ with $T(A) \in \mathcal{T}$ and $F(A) \in \mathcal{F}$.

In [1], the connection between torsion theories (in the pointed case), factorization systems, and absolute Galois structures was examined in detail.

More recently, in [2] a notion of torsion theory suitable for a context without a zero object was introduced. The core of this definition lies in introducing a class $\mathcal{Z}$ of objects that plays the role of the zero object. Using this class, one can define an ideal of morphisms $N_{\mathcal{Z}}$, consisting of all morphisms that factor through an object of $\mathcal{Z}$. This approach leads to the definition of $\mathcal{Z}$-kernels and $\mathcal{Z}$-cokernels, which in turn allow us to define $\mathcal{Z}$-exact sequences. A torsion theory in a non-pointed context is given by a pair of full replete subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{C}$ such that, defining $\mathcal{Z} \coloneqq \mathcal{T} \cap \mathcal{F}$, we have:

i) $\mathrm{Hom}(T, F) \subseteq N_{\mathcal{Z}}$ for every $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
ii) for every object $A \in \mathcal{C}$, there exists a $\mathcal{Z}$-exact sequence $T(A) \to A \to F(A)$.

The purpose of this talk is to provide an analogue of the results obtained in [1] for torsion theories in a non-pointed context. We say that $\mathcal{Z}$ is a *class of zero objects* of a category $\mathcal{C}$ if $\mathcal{Z}$ is a full posetal mono-coreflective subcategory of $\mathcal{C}$. For example, let $\mathcal{C}$ be a regular category with an initial object, and define $\mathcal{Z}$ as the full subcategory whose objects are the quotients of the initial object; then $\mathcal{Z}$ in $\mathcal{C}$ is a class of zero objects.

Given a category $\mathcal{C}$ equipped with a class $\mathcal{Z}$ of zero objects, we study torsion theories $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{T} \cap \mathcal{F} = \mathcal{Z}$. We show that several results from [1] continue to hold in this context. In particular, under certain assumptions (some of which are already known for the pointed case), there is a correspondence between torsion theories, factorization systems, and Galois structures. Furthermore, for such Galois structures, the corresponding torsion theory allows for a simple characterization of central extensions.

Interesting examples of such torsion theories can be found in MV-algebras, in Heyting algebras, and in the dual of elementary toposes.

References

[1] T. Everaert, M. Gran, *Protoadditive functors, derived torsion theories and homology*, J. Pure Appl. Algebra 219 n.8 (2015), 3629-3676
[2] A. Facchini, C. Finocchiaro, M.Gran, *Pretorsion theories in general categories*, J. Pure Appl. Algebra 225 n.2 (2021), 106503