A cylindrical Diaconescu's equivalence

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Motivation. In Grothendieck topos theory, the powerful *Diaconescu's equivalence*

$$\mathbf{Topos}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq J\text{-}\mathbf{Flat}(\mathcal{C}, \mathcal{E}) \tag{1}$$

relates geometric morphisms with codomain $\mathbf{Sh}(\mathcal{C}, J)$ with certain functors with domain $\mathcal{C}$. The textbook account of classifying topos theory (e.g., that found in [2], [7], [9], [10]), based on *syntactic categories*, relies on Diaconescu's equivalence to handle all the topos-theoretic details.

However, there are other categorical interpretations of first-order logic, such as *doctrine theory* in the sense of Lawvere [8], that prioritise a fibred approach. This talk presents a generalisation of (1) suitable for incorporating *relative*, or fibred, phenomena within topos theory – for instance, expositing a classifying topos theory for doctrines.

Background literature. Many versions of a fibred approach to Diaconescu's equivalence exist within the literature. In Diaconescu's paper [5], from which the equivalence (1) gets its name, the category $\mathcal{C}$ is replaced with an *internal category* of another topos.

The internal category approach has its limitations; notably, the fibred categories that can be considered must be *small* and *strictly functorial* (as oppposed to pseudo-functorial). For this reason, there has been a resurgence of interest in the earlier approach of Giraud [6] based on stacks, with recent work by Caramello and Bartoli [1], [3] generalising this treatment within the formalism of *relative topos theory* (see [4]).

Both Diaconescu's work [5], Giraud's [6] and that of Caramello and Zanfa [4] contain 'change of base' theorems. In all cases, the change of base is kept 'outside' of the equivalence. Our improvement over the previous relative versions of (1) is to place the change of base 'inside' the equivalence – this is achieved by considering *cylindrical 2-diagrams*.

Main result. We establish an equivalence between the following categories:

$$\mathbf{Topos}\left(\begin{array}{cc} \mathcal{F} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ \downarrow f , & \downarrow C_{\pi_P} \\ \mathcal{E} & \mathbf{Sh}(\mathcal{C}, J) \end{array}\right) \simeq \mathbf{RelMorph}\left(\begin{array}{cc} (\mathcal{C} \rtimes P, K) & (\mathcal{E} \rtimes \mathcal{F} / f^*, \tilde{J}_{\mathrm{can}}) \\ \downarrow \pi_P , & \downarrow \pi_{\mathcal{E}} \\ (\mathcal{C}, J) & (\mathcal{E}, J_{\mathrm{can}}) \end{array}\right),$$

objects:

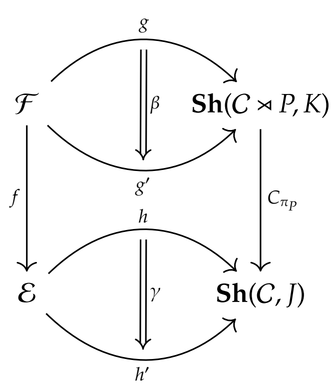
$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{g} & \mathbf{Sh}(\mathcal{C} \rtimes P, K) \\ f \downarrow & \cong & \downarrow C_{\pi_P} \\ \mathcal{E} & \xrightarrow{h} & \mathbf{Sh}(\mathcal{C}, J) \end{array}$$

commuting squares of geometric morphisms,

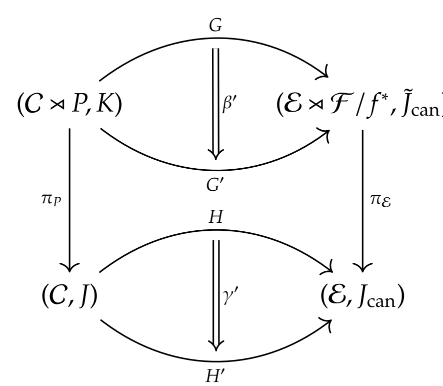
$$\begin{array}{ccc} (\mathcal{C} \rtimes P, K) & \longrightarrow & (\mathcal{E} \rtimes \mathcal{F} / f^*, \tilde{J}_{\mathrm{can}}) \\ \pi_P \downarrow & \cong & \downarrow \pi_{\mathcal{E}} \\ (\mathcal{C}, J) & \longrightarrow & (\mathcal{E}, J_{\mathrm{can}}) \end{array}$$

morphisms of relative sites,

arrows:



commuting cylindrical 2-diagrams of geometric morphisms (i.e. $\gamma * f \simeq C_{\pi_P} * \beta$),



commuting cylindrical 2-diagrams of functors (i.e. $\gamma' * \pi_P \simeq \pi_{\mathcal{E}} * \beta'$).

Here $\pi_P \colon \mathcal{C} \rtimes P \to \mathcal{C}$ denotes the *Street fibration* associated to a pseudo-functor $P \colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{CAT}$ (see [11], [12]).

We also discuss extensions of the notion of *subcanonical topology* to the relative setting, and applications to branches of categorical logic, such as doctrine theory. This work is adapted from the author's PhD thesis.

References

[1] L. Bartoli and O. Caramello, "On morphisms of relative toposes," 2023. arXiv: `2310.20691 [math.AG]`.

[2] O. Caramello, *Theories, sites, toposes: relating and studying mathematical theories through topos-theoretic 'bridges'*. Oxford University press, 2018.

[3] O. Caramello, "Fibred sites and existential toposes," 2022. arXiv: `2212.11693 [math.AG]`.

[4] O. Caramello and R. Zanfa, "Relative topos theory via stacks," 2021. arXiv: `2107.04417 [math.AG]`.

[5] R. Diaconescu, "Change of base for toposes with generators," *Journal of Pure and Applied Algebra*, vol. 6, no. 3, pp. 191–218, 1975.

[6] J. Giraud, "Classifying topos," in *Toposes, Algebraic Geometry and Logic*, F. W. Lawvere, Ed., Springer Berlin Heidelberg, 1972, pp. 43–56.

[7] P. T. Johnstone, *Sketches of an Elephant: A topos theoretic compendium*. Oxford University Press, 2002.

[8] F. W. Lawvere, "Adjointness in foundations," *Dialectica*, vol. 23, no. 3/4, pp. 281–296, 1969.

[9] S. MacLane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory* (Universitext). Springer New York, 1994.

[10] M. Makkai and G. E. Reyes, *First Order Categorical Logic: Model-Theoretical Methods in the Theory of Topoi and Related Categories* (Lecture Notes in Mathematics). Springer Berlin, Heidelberg, 1977.

[11] R. Street, "Fibrations in bicategories," *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, vol. 21, no. 2, pp. 111–160, 1980.

[12] R. Street, "Conspectus of variable categories," *Journal of Pure and Applied Algebra*, vol. 21, pp. 307–338, 1981.