Extensions and adjoint split extensions in the 2-category of adjunctions

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Categories and adjunctions between them form the 2-category LAdj, where the adjunctions go in the direction of the left adjoints. We restrict our attention to the sub 2-category $\mathsf{LAdj}_{\varnothing,*}$, whose objects are categories possessing initial and terminal objects. Since $\mathsf{LAdj}_{\varnothing,*}$ is 2-pointed (with the terminal category $\{*\}$ being 2-zero object), we can talk about extensions

$$\mathcal{X} \underset{i^R}{\overset{i}{\rightleftarrows}} \mathcal{A} \underset{p^R}{\overset{p}{\rightleftarrows}} \mathcal{B} \ ,$$

which are pairs adjunctions composing to the zero adjunction (the adjunction between the constantly initial and constantly terminal functors), satisfying an exactness condition. The exactness condition is close to saying that $\mathcal{X}$ and $\mathcal{B}$ (viewed as full subcategories) form a torsion theory in $\mathcal{A}$.

If the functor p additionally has a left adjoint s, then we talk of an adjoint split extension, and if $p^R = s$, we talk of a pointed adjoint split extension. It turns out that if we further restrict our attention to particularly nice categories, such as semi-abelian categories, or the opposites of toposes, as considered in [1], then adjoint split extensions are equivalent to ones that comes from a sort of a semi-direct product construction.

A nice example of this is the torsion theory, considered in [2], that comes from the pointed adjoint split extension

$$\mathsf{Lie}_{\mathbb{K}} \underset{i^R}{\overset{i}{\rightleftarrows}} \mathsf{ccHopf}_{\mathbb{K}} \begin{array}{c} \xleftarrow{\ s\ } \\ \perp \\ \xrightarrow{\ p\ } \\ \perp \\ \xleftarrow[\ s\ ]{} \end{array} \mathsf{Gp} \ .$$

We have that $\mathsf{ccHopf}_{\mathbb{K}} \simeq \mathsf{Lie}_{\mathbb{K}} \rtimes \mathsf{Gp}$, which is just the global formulation of the Cartier-Gabriel-Kostant theorem, which allows one to express a cocommutative Hopf algebra as the semi-direct product of a Lie algebra and a group.

References

[1] P.F. Faul, G. Manuell, Artin glueings of toposes as adjoint split extensions. *J. Pure Appl. Algebra* 227 (2023), no.5, Paper No. 107273, 40 pp.

[2] M. Gran, G. Kadjo, J. Vercruysse, A torsion theory in the category of cocommutative Hopf algebras. *Appl. Categ. Structures* 24 (2016), no.3, 269–282.

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