Composition and Recursion for Causal and Uniformly Continuous Structures

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Causality, a fundamental concept often associated with the dependency of present behaviour on past events, has been extensively studied in the context of functions on streams [HKR17]. More generally, causal morphisms can be understood as certain morphisms between coalgebras for an endofunctor $F\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ that has α^{op}-limits for some sufficiently large limit ordinal α. In our recent paper [BR23], we compared three definitions of causality, each shedding light on different facets of the concept. The first definition requires that F preserves α^{op}-limits, which allows the construction of a final coalgebra νF of F as limit of the diagram $\Phi F\colon \alpha^{\mathrm{op}} \to \mathcal{C}$ with $(\Phi F)_\beta = \lim_{\gamma<\beta} F((\Phi F)_\gamma)$. The diagram ΦF is called the *final chain* of F and we denote the limit projections by $p_\beta\colon \lim \Phi F \to (\Phi F)_\beta$. A morphism $f\colon \nu F \to \nu F$ is called *causal*, if for all $g, h\colon X \to \nu F$ and β the identity $p_\beta \circ g = p_\beta \circ h$ implies $p_\beta \circ f \circ g = p_\beta \circ f \circ h$ [RP19]. The second definition broadens the scope by considering causal morphisms as morphisms (natural transformations) in the category of α^{op}-diagrams. If $\mathcal{C}$ is Cartesian closed, then this framework introduces a versatile method for constructing causal morphisms via sequential and parallel composition and recursion, building on techniques from known as step-indexing [Bas19; Bir+12; Mon00] The third definition identifies causal maps as non-expansive maps in the category **Met** of metric spaces, if α is the first countable ordinal ω. This perspective enhances our comprehension of causality in specific scenarios, such as streams and partial computations.

The third characterisation of causal morphisms suggests that there may be generalisations of causality that correspond to other forms of continuity, since non-expansive maps are Lipschitz-continuous with Lipschitz constant 1. Thus, we wish to find variations of the first two characterisations that correspond to other forms of continuity, such as Lipschitz-continuity with other constant and uniform continuity, without going to general continuity [GHP09b; GHP09a].

In this talk, I will present our category theoretical framework for causal maps, which organises these into traced monoidal categories, and show how this framework can be extended to encompass forms of uniform continuity.

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