

Conditional expectations as split idempotents

Paolo Perrone*
University of Oxford

Based on joint work with Noé Ensarguet and Ruben Van Belle.

Conditional expectations are a cornerstone concept in probability theory. They are used to make mathematically precise the idea of “averaging over things we don’t know”, and so they are an essential tool in stochastic processes, where as time progresses, we gather more and more information about the state of our system. In particular, they are central to the theory of martingales.

Conditional expectations have a rich and regular mathematical structure, and in particular they are closed under composition. Despite these properties, and despite their conceptual importance, for decades they have been missing a categorical description.

Here we present one such description: in a dagger category of probability spaces and couplings, we show that regular conditionals correspond exactly to idempotent morphisms, which in this category are all dagger-split (in the sense of [4]). One leg of the splitting can be seen as the conditional distribution, and it acts functorially on random variables by taking their conditional expectation in a universal way.

This description seems to incorporate all known properties of conditional expectations, and to fully reflect its usage by the working probability theorist.

References

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*www.paoloperrone.org.