Welcome to the machine

a tale of bicategories, with a hint of British prog

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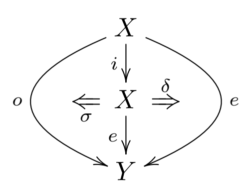
It has long been known [Ehrig et al., 1974] that automata can be interpreted within every monoidal category $(\mathcal{K}, \otimes, I)$; the cornerstone results in this direction are essentially three:

S1. if $T : \mathcal{K} \to \mathcal{K}$ is a commutative monad, Mealy and Moore machines in the (monoidal) Kleisli category $\mathcal{K}_T$ are 'non-deterministic' machines for a notion of fuzziness fixed by T;

S2. if $\mathcal{K}$ is closed, one can characterize Mealy and Moore machines coalgebraically [Jacobs, 2006], and in particular provide a slick proof of the cocompleteness of the categories $\mathbf{Mly}(A, B)$ and $\mathbf{Mre}(A, B)$ that they form [Adámek and Trnková, 1990];

S3. if (and *essentially only if*) $\mathcal{K}$ is Cartesian monoidal, $\mathbf{Mly}(A, B)$ is the hom-category of a bicategory **Mly** [Guitart, 1974, Katis et al., 1997], and $\mathbf{Mre}(A, B)$ the hom-category of a *semibicategory* (a bicategory without identity 1-cells, cf. [Boccali et al., 2023]) **Mre**.

Starting from the mantra that a monoidal category is nothing but a single-object bicategory, we fix a bicategory $\mathcal{B}$ and study 'abstract machines' in $\mathcal{B}$, i.e. diagrams of 2-cells of the form



where i, e, o are 1-cells respectively dubbed the 'input' 1-cell, the 'state' 1-cell and the 'output' 1-cell.

We then proceed to find parallels for S1, S2, S3 in this more general setting:

B1. let T be a monad on **Set** and $(V, \odot, \perp)$ a quantale. The study of bicategorical machines in the bicategory of (T, V)-relations of [Hofmann et al., 2014] accounts for notions of non-determinism that are modeled on topologies, approach structures, metric and ultrametric structures, Kuratowski closure spaces, and all the likes of structures studied by monoidal topology;

B2. in perfect parallel with the monoidal case, the *behaviour* of a Mealy/Moore machine can be characterized through a universal property [Goguen, 1972]; a terminal coalgebra for monoidal machines, a weighted limit of sorts for bicategorical machines. In the case of Moore machines the description is prettier, in terms of a right extension. This clarifies long-forgotten remarks by Bainbridge [Bainbridge, 1975] on abstract machines as Kan arrows;

B3. passing from single- to multi-object bicategories, we gain an additional degree of freedom indexing hom-categories over generic objects; in particular, we gain a rich compositional structure that was not present in the monoidal case, a way of composing machines that is neither sequential nor parallel and that we dub *intertwining*.

This is a joint work with A. Laretto, G. Boccali, S. Luneia, see arXiv:2303.03865.

References

[Adámek and Trnková, 1990] Adámek, J. and Trnková, V. (1990). *Automata and Algebras in Categories*. Kluwer.

[Bainbridge, 1975] Bainbridge, E. (1975). Addressed machines and duality. In Manes, E. G., editor, *Category Theory Applied to Computation and Control*, pages 93–98, Berlin, Heidelberg. Springer Berlin Heidelberg.

[Boccali et al., 2023] Boccali, G., Femić, B., Laretto, A., Loregian, F., and Luneia, S. (2023). The semibicategory of moore automata.

[Ehrig et al., 1974] Ehrig, H., Kiermeier, K.-D., Kreowski, H.-J., and Kühnel, W. (1974). *Universal theory of automata. A categorical approach.* Teubner Studienbücher. Informatik. Stuttgart.

[Goguen, 1972] Goguen, J. (1972). Realization is universal. *Mathematical systems theory*, 6:359–374.

[Guitart, 1974] Guitart, R. (1974). Remarques sur les machines et les structures. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 15:113–144.

[Hofmann et al., 2014] Hofmann, D., Seal, G., and Tholen, W. (2014). *Monoidal Topology: A Categorical Approach to Order, Metric, and Topology*, volume 153. Cambridge University Press.

[Jacobs, 2006] Jacobs, B. (2006). A bialgebraic review of deterministic automata, regular expressions and languages. In *Algebra, Meaning, and Computation*, pages 375–404. Springer Berlin Heidelberg.

[Katis et al., 1997] Katis, P., Sabadini, N., and Walters, R. (1997). Bicategories of processes. *Journal of Pure and Applied Algebra*, 115(2):141–178.