The homotopy posets of a category

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We introduce invariants which, to a category C, associate functors

$$\pi_0(C/\_)\colon C \to \mathbf{Pos}_\bullet, \qquad \pi_1(C/\_)\colon C \to \mathbf{Pos}_\bullet$$

with values in the category of pointed posets. We call these the *zeroth and first homotopy posets* of a category. These generalise the π_0 and π_1 of a groupoid in the following sense.

Theorem 0.1 — *Let G be a groupoid and x an object in G. Then*

1. *$\pi_0(G/_x)$ is a discrete poset, isomorphic to the set $\pi_0(G)$ of connected components of G, pointed with the connected component of x,*
2. *$\pi_1(G/_x)$ is a discrete poset, isomorphic to the underlying pointed set of the group $\pi_1(G, x) = \mathrm{Hom}_G(x, x)$.*

Their main property is that they "vanish" precisely on terminal objects in C.

Theorem 0.2 — *Let C be a category and x an object in C. The following are equivalent:*

(a) $\pi_0(C/_x) = \{[x]\}$ and $\pi_1(C/_x) = \{[x]\}$,

(b) x is a terminal object in C.

More precisely, π_0 vanishes on weak terminal objects and π_1 on subterminal objects.

In this sense, any element of $\pi_0(C/_x)$ and $\pi_1(C/_x)$ besides the basepoint may be seen as an "obstruction to x being a terminal object". This gives a structured answer to the question: *how far is an object from being terminal?* Via the identification of isomorphisms $f\colon y \to x$ with terminal objects in the slice category $C/_x$, this in turn provides a structured answer to the question: *how far is a morphism from being an isomorphism?*

In my talk, after giving the definition of the invariants, I will explore some of their properties more in depth: in what ways they are functorial, and under what conditions the homotopy posets are endowed with extra properties or structures, such as the existence of joins or monoidal structures.

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