Quotients, Extensionality and Choice in Relational Doctrines *

Francesco Dagnino

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Among categorical approaches to logic, Lawvere's doctrines [3, 4] stand out as a simple and powerful framework capable to cope with a large variety of situations. Doctrines are functors $P : \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$, on a category $\mathcal{C}$ with finite products, providing an algebraic description of theories in predicate logic: objects and arrows of $\mathcal{C}$ model contexts and terms, products of $\mathcal{C}$ model context concatenation and P maps each object to a poset that models formulas on that object, ordered by logical entailment. Doctrines have proved very effective for studying important logical notions such as quotients, extensional equality and rules of choice as well as related universal constructions [6, 7].

A longstanding variable-free alternative to predicate logic is the *calculus of relations* [1, 8, 9]. Here one takes as primitive concepts (binary) relations instead of (unary) predicates, with some basic operations, such as relational identities, composition and converse. However natural, this calculus has been much less studied using functorial tools. In this talk we introduce *relational doctrines* as a functorial description of the calculus of relations. We show that definitions of quotient, extensional equality and rules of choice smoothly extend to relational doctrines as well as the associated universal constructions. Then, thanks to the variable-free nature of the calculus of relations, we can get rid of products in the base category, thus recovering many new instances of the presented constructions, such as, the exact completion of a category with weak finite limits and categories of metric structures. Moreover, we will show by examples how these notions can capture well-known metric and topological properties, such as separation in metric and topological spaces, Cauchy-completeness [5], and compactness of topological spaces [2].

*This is joint work with Fabio Pasquali

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