# A conservativity-like result for a propositional type theory

Matteo Spadetto
University of Leeds
matteo.spadetto.42@gmail.com

A dependent type theory is said to have *propositional identity types* if it is endowed with a type constructor satisfying the usual formation, introduction and elimination rules of intensional identity types, but not the corresponding computation rule, which is only required to hold in a weakened form, called *propositional form*. In detail, as illustrated in [4], whenever we are given judgements:

$$x, y : A; p : x = y \vdash C(x, y, p) : \textsc{Type}$$
$$x : A \vdash q(x) : C(x, x, r(x))$$

in place of asking that the judgemental equality $x : A \vdash J(x, x, r(x), q) \equiv q(x)$ holds, we only ask that it holds propositionally, i.e. that the type:

$$x : A \vdash J(x, x, r(x), q) = q(x)$$

is inhabited (here $J$ denotes the identity type eliminator). One might consider the same form of weakening for the computation rule of dependent sum types and dependent product types: these type constructors satisfying a propositional computation rule will be called *propositional dependent sum types* and *propositional dependent product types* respectively.

In this talk we consider a dependent type theory having propositional identity types, propositional dependent sum types and propositional dependent product types, together with an arbitrary family of basic types, and call it *propositional type theory*. If $|-|$ is the canonical interpretation of the propositional type theory into the extensional type theory, we identify a concrete family $\mathcal{F}$ of type judgements $\Gamma \vdash T : \textsc{Type}$ of the propositional type theory such that the extensional type theory is conservative over the propositional one *relatively* to $\mathcal{F}$. In other words, whenever the extensional type theory infers a judgement of the form $|\Gamma| \vdash t : |T|$ for some judgement $\Gamma \vdash T : \textsc{Type}$ of $\mathcal{F}$, then the propositional type theory infers a judgement of the form $\Gamma \vdash \bar{t} : T$.

Therefore, despite this non-negligible (and actual) weakening of the propositional type theory with respect to the extensional one, there is actually an interesting family of judgements in which the two theories have the same deductive power. This result is obtained by re-adapting a proof strategy contained in [2] and exploits a classic notion of sematics for dependent type theories described in [1, 3] and based on the notion of *category with attributes*.

## References

[1] John Cartmell. *Generalised Algebraic Theories and Contextual Categories*. PhD thesis, University of Oxford, 1978.

[2] Martin Hofmann. Conservativity of equality reflection over intensional type theory. In Stefano Berardi and Mario Coppo, editors, *Types for Proofs and Programs*, pages 153–164, Berlin, Heidelberg, 1996. Springer Berlin Heidelberg.

[3] Eugenio Moggi. A category-theoretic account of program modules. *Mathematical Structures in Computer Science*, 1(1):103–139, 1991.

[4] Benno van den Berg. Path categories and propositional identity types. *ACM Trans. Comput. Logic*, 19(2), jun 2018.