Lax normal conical 2-limits and the Grothendieck construction

As it is usually understood, when one moves to dimension 2, or in general to an enriched setting, there is the need to consider weighted limits, as conical limits do not suffice anymore. However, for 2-categories, weighted 2-limits are not the only solution to the problem. There is indeed another idea that might come up: relaxing the concept of cone to admit coherent 2-cells inside. After briefly recalling the definition of weighted enriched limit, we will describe the problem of conicalizing weighted 2-limits. Such problem has, strictly speaking, no solution. But we will search for some essential solution to it, accepting the cones to have 2-cells inside.

Our wish to find an essential solution to the problem of conicalizing weighted 2-limits will simultaneously create an extension of the Grothendieck construction and the relaxed notion of conical 2-limit that we need. We call the latter lax normal conical 2-limit, and show that every weighted 2-limit can be reduced to one of this form. The process itself will justify the Grothendieck construction and reveal its bond with the laxness.

We will then study the extended Grothendieck construction and the corresponding extended notion of fibration from a more abstract point of view. But the laxness that permeates the Grothendieck construction will force us to move out of 2-$\mathcal{CAT}$, in order to consider lax natural transformations.

We will see that the pointwise Kan extensions are the key to the meaning of the Grothendieck construction. But there is no known definition of pointwise Kan extension in this weaker version of 2-$\mathcal{CAT}$, and so we will firstly propose one. A generalized lax but not too lax version of the parametrized Yoneda lemma will help us with this, along with offering a better understanding of what it means to be lax normal.