A few remarks of the fibration of algebras

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Our aim is to study particular instances of split fibrations and opfibrations arising in the following fashion: let F_A be a family of endofunctors of a category $\mathcal{X}$, depending functorially on a parameter $A \in \mathcal{A}$ living in a possibly different category of indices $\mathcal{A}$, so that a morphism $A \to A'$ induces a natural transformation $F_A \Rightarrow F_{A'}$ (or in the opposite direction). Then the category $\mathsf{Alg}(F_A)$ of endofunctor algebras of F_A is the typical fiber of a fibration over $\mathcal{A}$, the *fibration of algebras* induced by a functor $F_\bullet : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]$. If each F_A is pointed (resp., a monad) one can consider pointed algebras (resp., Eilenberg-Moore algebras, or free algebras); dually if each F_A is copointed, or a comonad. Examples of this construction are 'everywhere' and we aim to offer a unified perspective on the matter:

- consider the "action monad" $T_M = - \times M$ indexed by a monoid M: the fibration is the so-called fibration of modules, well-known to algebraic geometers and homotopy theorists;
- consider the "writer" comonad $S_I = - \times I$ indexed by a set I: the fibration glueing together all coKleisli categories of S_I is the simple fibration, and old friend of type theorists; similarly, the coEilenberg-Moore fibration of S_I is the collection of all 'simple slices';
- when the indexing category is the arrow category of a pretopos $\mathcal{E}$, consider a polynomial endofunctor P_f depending on $f \in \mathcal{E}^{\to}$: the fibration of algebras of P_f represents operations on objects in the pretopos;
- when the indexing category is the lattice $\mathcal{L}$ of *levels* of a sheaf topos $\mathcal{B}$ we can describe intrinsically Lawvere's theory of essential localizations: this fibration of algebras is now an 'étalé space' of some sort over $\mathcal{L}$.

This is a work in progress with Davide Castelnovo and Greta Coraglia.