SUMMABILITY IN ALGEBRAS OF GENERALIZED POWER SERIES AND BEYOND

PIETRO FRENI

Abstract: Spaces of generalized power series have been important objects in asymptotic analysis and in the algebra and model theory of valued structures ever since the introduction of the first instances of them by Levi-Civita and Hahn. A space of generalized series can be understood to be built up from a triple $(\Gamma, <, \mathcal{F})$ where $(\Gamma, <)$ is an ordered set and $\mathcal{F}$ is an ideal of Noetherian subsets of Γ, as the $\mathbf{k}$-vector space $\mathbf{k}(\Gamma, \mathcal{F})$ of functions $f : \Gamma \to \mathbf{k}$ whose support $\operatorname{Supp} f = \{\gamma \in \Gamma : f(\gamma) \neq 0\}$ lies in $\mathcal{F}$. A prominent feature of these objects is the notion of formal infinite sum, or rather more appropriately, of *infinite* $\mathbf{k}$*-linear combination*: a family $(f_i)_{i \in I} \in \mathbf{k}(\Gamma, \mathcal{F})$ such that $\bigcup_{i \in I} \operatorname{Supp} f_i \in \mathcal{F}$ and for every $\gamma \in \Gamma$ the set $\{i \in I : \gamma \in \operatorname{Supp} f_i\}$ is finite, is said to be *summable*. Given such a family it is possible to associate to any $(k_i)_{i \in I} \in \mathbf{k}^I$ the infinite linear combination $f = \sum_{i \in I} f_i k_i$ given by $f(\gamma) = \sum (f_i(\gamma) : \gamma \notin \operatorname{Supp}(f_i))$. This feature is referred throughout literature as a *strong linear* structure on the space $\mathbf{k}(\Gamma, \mathcal{F})$ and maps $F : \mathbf{k}(\Gamma, \mathcal{F}) \to \mathbf{k}(\Delta, \mathcal{G})$ preserving it, are referred to as *strong(ly) linear maps* (cfr e.g. [1], [2]).

In this talk we will argue about a suitable category-theoretical framework for the study of the above outlined notion of strong linearity, in particular we will justify a notion of *reasonable category of strong* $\mathbf{k}$*-vector spaces* generalizing the above setting and prove that up to equivalence there is a unique universal category $\Sigma\mathrm{Vect}$ with the property that every reasonable category of strong vector spaces has a fully faithful functor to $\Sigma\mathrm{Vect}$. We will see $\Sigma\mathrm{Vect}$ can be defined as an orthogonal subcategory of Ind-($\mathrm{Vect}^{\mathrm{op}}$). We will describe a monoidal closed structure for $\Sigma\mathrm{Vect}$ and the relation $\Sigma\mathrm{Vect}$ has with another orthogonal subcategory of Ind-($\mathrm{Vect}^{\mathrm{op}}$) equivalent to the category of linearly topologized vector spaces that are colimits of linearly compact spaces.

Finally we will present some open questions in this setting.

References

[1] Alessandro Berarducci and Vincenzo Mantova. Surreal numbers, derivations and transseries. *Journal of the European Mathematical Society*, (2):339–390, jan.

[2] Joris van der Hoeven. Operators on generalized power series. *Illinois Journal of Mathematics*, 45(4):1161 – 1190, 2001.