Internal pseudomonoidal structures and Day convolution in monoidal 2-categories

Sophie d'Espalungue

Abstract

When $\mathcal{C}$ is a monoidal category, the category of functors $[\mathcal{C}, Sets]$ inherits a monoidal structure via the Day convolution product. As a matter of fact, this monoidal structure is often considered to be induced on the $\mathcal{V}$-enriched category of $\mathcal{V}$-enriched functors $[\mathcal{C}, \mathcal{V}]_{\mathcal{V}}$ with $\mathcal{C}$ a $\mathcal{V}$-enriched monoidal category, turning $[\mathcal{C}, \mathcal{V}]_{\mathcal{V}}$ into a $\mathcal{V}$-enriched monoidal category. Pseudomonoids can be defined internally into any monoidal 2-category, as well as monoids internal to a pseudomonoid, encompassing the notions of monoidal category and $\mathcal{V}$-enriched monoidal category as pseudomonoids internal to $(\textsc{cat}, \times)$ or $(\textsc{cat}_{\mathcal{V}}, \times_{\mathcal{V}})$ with their internal monoids. The purpose of this presentation is to make precise the conditions under which one can generalize the Day convolution product for pseudomonoids internal to some closed monoidal 2-category, especially for a monoidal structure that is not necessarily the one for which this 2-category is closed.

As an example and an application, I will briefly talk about one particular situation where the need of such a generalization occured, by considering this construction in the cartesian closed monoidal 2-category of sequences of categories $(\textsc{cat}^{\mathbb{N}}, \times, [-,-]_{\mathbb{N}})$, with respect to the monoidal structure given by operadic composition $\circ$ of sequences - so that pseudomonoids with respect to $\circ$ are $\textsc{cat}$-valued (pseudo-)operads, whose internal monoids coincide with Batanin's definition (1) of operads internal to a categorical operad. We will therefore be able to equip the sequence of presheaves $[\mathcal{P}^{op}, Sets]_{\mathbb{N}}(n) = [\mathcal{P}^{op}(n), Sets]$ over a $\textsc{cat}$-valued operad $\mathcal{P}$ with the structure of a pseudo-operad. When $\mathcal{P}$ is the associahedral operad Υ (whose algebras are $\mathcal{A}_\infty$-monoidal categories - see (2)), considering operads internal to its pseudo-operad of presheaves $[\Upsilon^{op}, Sets]_{\mathbb{N}}$ will provide a nicely appropriate framework which will yield lots of new tools for the study of homotopy associative structures.

References

[1] Batanin, M. A. The eckman-hilton argument and higher operads. *Adv. Math. (217)(2008), pp. 334-385.*.

[2] Fiedorowicz, Z., Gubkin, S., and Vogt, R. M. Associahedra and weak monoidal structures on categories. *Algebr. Geom. Topol. 12* (2012), 469–492.