Relational doctrines, monads and lax algebras

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Lawvere's hyperdoctrines (or simply doctrines) [7, 8] provide a fairly simple and intuitive categorical framework for studying severl kinds of logics. Indeed, a doctrine is just a functor $P : \mathcal{C}^{\mathrm{op}} \to \mathcal{P}os$ where each poset $P(X)$ over an object X in $\mathcal{C}$ collects predicates over the object X ordered by logical entailment. This notion nicely captures the usual view of logics, having predicates as the primitive concept, and then it allows to study more expressive systems just by adding (algebraic) structure on doctrines.

Another relevant approach to logic, dating back to Tarski's analysis of the calculus of relations [10], takes relations as the primitive concept. This provides a valid variable-free alternative to usual logics and it is of increasing interest both for computer scientists and mathematicians, as relational structures are ubiquitous and moreover they seem to be more suited to deal with new computaitonal models. Many categorical models have been proposed to distill the essence of relations, such as allegories [5] and cartesian bicateogires [3].

In this talk, we will show how the language of doctrines can be adapted to support logics based on relations. We will introduce the 2-category of *relaitonal doctrines*, that is, doctrines having (binary) relations as primitive concept together with their essential algebraic structure: composition, identities and converse. The 1-arrows between such doctrines correspond exactly to relational extensions of functors, while 2-arrows are natural transformations compatible with such relaitonal extensions.

Then, we will consider monads in such 2-cateogry, showing how the construction of the category of lax algebras described in [6] can be generalised to monad on a relational doctrine. The resulting doctrine of lax algebras has a universal property similar to that of Eilenberg-Moore objects [9]. Instances of such construction are doctrines over preordered sets, metric spaces and topological spaces, thus showing their algebraic nature in a relaitonal setting.

Finally, we will show how to construct relaitonal doctrines from sufficiently structure doctrines. Notably, we do this for primary linear doctrines [4] which are both existential and elementary, that is, doctrines modelling the $\langle \otimes, \mathbf{1} \rangle$-fragment of linear logic enriched with usual equality and existential quantifiers. This generalises a similar result [2] yielding a cartesian bicategory from a non-linear existential elementary doctrine. Moreover, the resulting construction turns out to be 2-functorial and, when applied it to certain 1-arrows on doctrines of weak subobjects, it returns the well-know Barr extension of functors [1].

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