Algebras of the Giry monad

Ruben Van Belle

University of Oxford

In this talk, we explore the classification and representation of algebras of the Giry monad. Intuitively, these algebras define spaces where integration is well-defined. The structure map of the algebra assigns to each probability measure its *centre of mass*. However, characterizing these spaces for a given probability monad is nontrivial. For certain probability monads, representations of the algebras are well understood. Convex compact subsets of locally convex topological vector spaces correspond to algebras of the Radon monad [3], while convex closed subsets of Banach spaces characterize those of the Kantorovich monad [2]. The distribution monad's algebras can be represented as abstract convex sets [1, 4], but classifying the algebras of the Giry monad is more challenging.

For every abstract convex set C, there exists a canonical vector space V_C with a convex map $\phi_1 : C \to V_C$ [1, 4]. Similarly, we construct a functor $V_{(-)} : \mathbf{Mble}^G \to \mathbf{LCTVS}$, which assigns to every Giry algebra a locally convex topological vector space. Additionally, for every Giry algebra A there exists a structure-preserving map $\phi_2 : A \to V_A$.

A key difficulty in classifying Giry algebras is that the algebras of the Giry monad and the distribution monad do not always satisfy the *cancellation property* (C1), which is equivalent to ϕ_1 being an injection [4]. We introduce a *generalized cancellation property* (C2), which is analogously equivalent to ϕ_2 being an embedding. We endow Giry algebras with a canonical topology that makes the structure map continuous. By equipping the algebras with a topology, we prove that those satisfying (C2) can be represented as relatively closed convex subsets of locally convex topological vector spaces. Furthermore, in this setting, the structure map can be described as a *Pettis integral*, allowing the application of functional analysis techniques to general Giry algebras that satisfy (C2). This leads to an adjunction between the algebras of the Radon monad and the Giry algebras that satisfy (C2),

$$\mathbf{CH}^R \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Mble}^G_{C2}$$

For Giry algebras that do not satisfy (C2), we consider those whose associated vector space is finite dimensional. Under this assumption, we show that the conditions (C1) and (C2) are equivalent. We identify specific elements within the algebra as '*infinite elements*' and introduce a partial order on those elements. Every infinite element corresponds, in an order-preserving way, to a well-behaved subalgebra, where the structure map can be described using Pettis integrals. This provides an algorithmic approach to describing the structure map and the Giry algebra using methods from functional analysis and order theory.

It is not always the case that (C1) and (C2) are equivalent, which we demonstrate by providing an example of an algebra that does satisfy (C1) but not the generalized cancellation property (C2). We conclude with some potential applications for random variables taking values in a Giry algebra.

References

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